The Theory of Electromagnetic Field

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Mir Publishers · Moscow

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The Greek Alphabet

Αα	Alpha	Ιι	Iota	Ρρ	Rho
Ββ	Beta	Κκ	Kappa	Σσ	Sigma
Γγ	Gamma	Λλ	Lambda	Ττ	Tau
Δδ	Delta	Μμ	Mu	Υυ	Upsilon
Εε	Epsilon	Νν	Nu	Φφ	Phi
Ζζ	Zeta	Ξξ	Xi	Χχ	Chi
Ηη	Eta	Οο	Omicron	Ψψ	Psi
Θϑθ	Theta	Ππ	Pi	Ωω	Omega

The Russian Alphabet and Transliteration

Аа	a	Кк	k	Хх	kh
Бб	b	Лл	l	Цц	ts
Вв	v	Мм	m	Чч	ch
Гг	g	Нн	n	Шш	sh
Дд	d	Оо	o	Щщ	shch
Ее	e, ye	Пп	p	ъ	"
Ёё	e, ye	Рр	r	Ыы	y
Жж	zh	Сс	s	ь	'
Зз	z	Тт	t	Ээ	e
Ии	i	Уу	u	Юю	yu
Йй	y	Фф	f	Яя	ya

На английском языке

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PREFACE

The theory of an electromagnetic field is a concluding subject in the curriculum of the theoretical training of electrical engineers. The field theory course usually follows the course in the basic theory of electric circuits.

The author covers the basic aspects of the theory of an electromagnetic field within the restricted space so that the text can fit the interest of students and answer the practical engineering needs. The emphasis therefore is on the behavior of ferromagnetic bodies in an electromagnetic field and on the processes in real imperfect dielectrics and poor conductors. The text also gives strict definitions of the basic vectors and shows the differences between them.

Many of the aspects discussed in the book focus on the problems one has to deal with in electrical engineering practice. This is one of the factors that influenced the organization of the book and the manner of presentation of the material. A similar treatment of the field theory was also given in the book by A. Netushil and K. Polivanov *The Theory of Electromagnetic Field. Theoretical Principles of Electrical Engineering.* Part 3 (Gosenergoizdat, in Russian). No effort has been made to cite a great number of contributors in this area. In the author's opinion, of many valuable texts, the following books will be very helpful to students in the study of the theory of electrical engineering: *Analysis and Synthesis of Electric Circuits* (Mir Publishers) by G. Zeveke, P. Ionkin, A. Netushil, and S. Strakhov, *The Feynman Lectures on Physics* (Addison-Wesley) by R. Feynman, R. Leighton, and M. Sands, *Electricity and Magnetism* (McGraw-Hill) by E. Purcell that treats the effects of moving bodies in an electromagnetic field, and *Fundamentals of the Theory of Electricity* (Mir Publishers) by I. Tamm. Special mention should be made of Maxwell's *Treatise on Electricity and Magnetism* which forms the basis of the course in the field theory.

In preparing the book for its publication in English, the author revised the text and added some useful information.

Chapter 1 MAXWELL'S EQUATIONS

In his great work *Treatise of Electricity and Magnetism*, published in 1873, James Clerk Maxwell generalized the experimental and theoretical studies of Coulomb, Poisson, Laplace, Green, Gauss, Ampère, Faraday, Ohm, Lenz, and other scientists of the 18th and 19th centuries. Maxwell introduced a new electromagnetic theory. He showed that it was quite possible to predict and quantitatively describe electromagnetic phenomena from the concept of an electromagnetic field using a set of equations. The electromagnetic field equations introduced by Maxwell are known as Maxwell's equations.

The text presupposes that the reader is familiar with Maxwell's equations, in any case, in their integral form. Also, it is expected that the reader knows the definitions of all the quantities appearing in these equations.

1.1. Classical Electrodynamics

Considering the interaction of an electromagnetic field with a substance, it often proves convenient to limit the discussion to a macroscopic behavior, i.e. regard a material medium as being continuous and characterize it by averaged values. The theory of field that uses the macroscopic concept of a continuum is called *macroscopic theory*. The theory of an electromagnetic field is often referred to as *electrodynamics*. If it considers the macroscopic properties of media, the theory is said to be the electrodynamics of continuous media, or macroscopic electrodynamics.

A further development in the theory of electromagnetism relies on the microscopic concept of substances.

The laws of classical physics are applicable to the electrodynamics of continuous media, whether stationary or slowly moving. For this reason, the electromagnetic theory presented here can be called *classical electrodynamics*. This book describes an electromagnetic field in the absence of moving bodies, although a considerable part of the most important electrical engineering devices and apparatus depend for their operation precisely on the motion of conductors and dielectrics in a magnetic or electric field. These are electrical machines such as electric motors and almost all generator types, magnetohydrodynamic (MHD) devices operating on the principle of interaction between magnetic fields and electrically conductive fluids and gases (plasmas), numerous devices relying on the controlled field of streams of electrons, ions, and other charged particles, etc.

Classical electrodynamics essentially agrees with Maxwell's theory, though the basic statements specified by the equations in his famous treatise were extended and

substantially new conclusions were made, the most important one being the radiation of electromagnetic waves. It should also be noted that after the fundamental studies of Lorentz and the birth of electronic theory, the viewpoint on the interaction of a field with polarized material media has changed and now differs in many respects from that introduced by Maxwell. This explains why the classical field theory is sometimes called Maxwell-Lorentz electrodynamics.

Circuit theory and field theory. Almost all electromagnetic processes occurring in electrical equipment can be considered from the viewpoint of field theory. On the other hand, it is expedient to separate the theory of electric circuits from the general field theory because there is a large number of specific laws and calculation techniques which make it unnecessary to refer constantly to the theory of field. However, the physical concepts the circuit theory has to deal with, like many of its laws, follow exactly from the field theory. The analytical expressions for electric circuit parameters r, C, L, and M can only be derived proceeding from the field theory.

Many branches of applied electrical engineering are even more closely related to the field theory than to the theory of circuits. To these belong not only the basic branches of radio engineering dealing with radio wave propagation and reception, but also many branches of the theory of electrical machines and generators, including the theoretical aspects of insulation, high-voltage generation, etc.

1.2. Maxwell's Equations in Integral Form

Maxwell's first equation is nothing else than the known circuital law (Ampère's law) or Maxwell-Ampère's equation:

$$\oint \mathbf{H} d\mathbf{l} = \int \mathbf{J}_d \mathbf{S} \tag{1.1}$$

Its right side gives the total current, i.e. the sum of the current of density $\mathbf{J}$ due to the flow of electric charges (commonly, it is just the conduction current) and the displacement current whose density is equal to the time derivative of electric flux density (electric displacement density):

$$\mathbf{J}_d = \partial \mathbf{D} / \partial t$$

The integral on the left side is the closed-loop, or line integral (the integral along the closed-path line l) of a magnetic field strength $\mathbf{H}$; for vacuum, the vector $\mathbf{H}$ is equal to the magnetic induction, or magnetic flux density $\mathbf{B}$, divided by the magnetic constant: $\mathbf{H} = \mathbf{B}/\mu_0$.

The integral on the right side is the surface integral of the total current density. This integral is taken over any surface bounded by contour l. In the simplest case, Eq. (1.1) has the form

$$\oint \mathbf{H} d\mathbf{l} = \Sigma i w \tag{1.1a}$$

Maxwell's second equation[1] is the known law of electromagnetic induction, also called Maxwell-Faraday's equation. This law states that the emf $\mathscr{E}$ induced in the

[1] We have numbered Maxwell's equations here as adopted in the electrical engineering literature. Other books might not number Maxwell's equations at all or number them differently from this text.

closed path by a changing magnetic field is equal to the time rate of decrease of the magnetic flux linking the loop. Defining the emf as the work done in moving a unit charge around a closed path, we may represent the emf (voltage) by an integral, $\mathscr{E} = \oint \mathbf{E} d\mathbf{l}$, since **E** is the force on the unit charge. According to Maxwell's second equation, this integral is related to the derivative of magnetic flux $\Phi = \int \mathbf{B} d\mathbf{S}$ by the equality

$$\mathscr{E} = \oint \mathbf{E} d\mathbf{l} = (-\partial/\partial t) \int \mathbf{B} d\mathbf{S} \tag{1.2}$$

The next equation out of the set of Maxwell's equations is known as *the Gauss theorem* for electrostatics (for electric charge)

$$\Psi = \oint \mathbf{D} d\mathbf{S} = Q = \int \rho dV \tag{1.3}$$

In words, this law tells us that the outward flux Ψ of density **D** through a closed surface S is equal to the charge enclosed within the surface. The charge Q is expressed as the volume integral of the charge density ρ; the integral is certainly taken over the entire volume V bounded by the surface S in question.

For a vacuum, the electric flux density **D** and electric field strength **E** differ just formally by the electric constant ε_0:

$$\mathbf{D} = \varepsilon_0 \mathbf{E}$$

Equation (1.3) finds wide use in the calculation of electrostatic fields. Maxwell showed that this equation is also applicable to time-varying fields.

A similar equation defining a magnetic flux (i.e. magnetic induction **B**) says that the flux through any closed surface is equal to zero:

$$\oint \mathbf{B} d\mathbf{S} = 0 \tag{1.4}$$

In the simplest case, there are the following relations between the vectors entering into the equations given above:

$$\mathbf{J} = \sigma \mathbf{E}, \mathbf{D} = \varepsilon \varepsilon_0 \mathbf{E}, \mathbf{B} = \mu \mu_0 \mathbf{H} \tag{1.5}$$

The first relation expresses Ohm's law in vector form. The coefficients σ, ε, μ are the parameters characterizing a medium. The quantity σ is called *conductivity*. In the physical literature, the quantities ε and μ are termed the *dielectric permittivity* and *permeability* respectively. In the electrical engineering literature, they are called the *relative permittivity* (or *dielectric constant*) and *relative permeability* respectively, with the subscript r attached to each symbol: ε_r and μ_r. Throughout this book, we will designate these quantities as ε and μ and call them permittivity and permeability for simplicity. The electric constant ε_0 and magnetic constant μ_0 depend on the choice of the basic systems of units and the adopted notation. In the SI system,

$$\varepsilon_0 = 8.856 \times 10^{-12} \text{ F/m and } \mu_0 = 4\pi \times 10^{-7} \text{ H/m}$$

They are also known as the permittivity and permeability of vacuum (free space), but these names that arose in the course of development of science have no physical meaning.

The relations between **D** and **E** and also between **H** and **B** will be considered in detail in Chs. 3 and 4.

1.3. Principal Electric Field Vector

The electric field strength **E**, or more briefly electric field, is found from the force **F** acting or capable of acting on a test charge q:

$$\mathbf{F} = q\mathbf{E} \tag{1.6}$$

If a test charge can disturb the field acting on this charge, the following expression applies:

$$\mathbf{E} = \lim_{q \to 0} (\mathbf{F}/q) \tag{1.7}$$

The above expressions in conjunction with Coulomb's law for charges in vacuum enable us to formulate this law in the field theory language: the charge $q = q_1$ has a field whose strength at a point lying a distance $\mathbf{R} = R\mathbf{e}_R$ away from the charge is

$$\mathbf{E} = (q/4\pi\varepsilon_0 R^2)\mathbf{e}_R \tag{1.8}$$

Potential and its gradient. The theory of circuits widely uses the notion of voltage defined as the integral of field strength and given as the potential difference:

$$u_{12} = \int_{1\text{-}2} \mathbf{E}d\mathbf{l} = \varphi_1 - \varphi_2 \tag{1.9}$$

We can identify voltage with potential difference if the **E** field is equal to minus the potential gradient [see Eq. (1.13)]:

$$\mathbf{E} = -\operatorname{grad}\varphi \tag{1.10}$$

The field of static charges is a potential field. Expression (1.10) is always valid if the electric field is related to electric charges according to Coulomb's law. Where this is the case, we can define Coulomb's law itself not for the field (1.8) but for a potential: for a point charge q in the absence of polarizable media

$$\varphi = q/4\pi\varepsilon_0 R \tag{1.11}$$

Taking the gradient of the last expression, observing (1.10), we directly arrive at (1.8) for a point charge q in the absence of polarized media.

Gradient. Let a scalar φ be a continuous differentiable function of space coordinates, $\varphi(x, y, z)$.

In moving it by

$$d\mathbf{s} = \mathbf{e}_x dx + \mathbf{e}_y dy + \mathbf{e}_z dz$$

the scalar φ grows by $d\varphi$ definable by the rules of differential calculus:

$$d\varphi = (\partial\varphi/\partial x)dx + (\partial\varphi/\partial y)dy + (\partial\varphi/\partial z)dz \tag{1.12}$$

It is worthy of note that the last expression can be regarded as a scalar product of vector $d\mathbf{s}$ with components dx, dy, dz and the vector having components $\partial\varphi/\partial x$, $\partial\varphi/\partial y$, $\partial\varphi/\partial z$. Indeed, Eq. (1.12) gives the sum of the products of like components of these two vectors. The second vector is called the *gradient*

$$\operatorname{grad}\varphi = (\partial\varphi/\partial x)\mathbf{e}_x + (\partial\varphi/\partial y)\mathbf{e}_y + (\partial\varphi/\partial z)\mathbf{e}_z \tag{1.13}$$

It is obvious that, with ds components being equal, the differential is at a max-

imum if $d\mathbf{s}$ is in the direction of grad φ. We thus can give such a definition of the gradient φ: this is a vector any ith component of which is equal to the rate of growth of φ in the direction of the ith coordinate; the direction of the gradient φ is the direction in which φ rises fastest. This definition permits us to express the gradient in any system of coordinates (see Appendix A5).

Graphic representation (mapping) of a field. The representation of an electric field by the lines of vector **E** (field lines or lines of force) and by the paths of an equal potential (equipotentials) aids in visualizing the **E** field. Representing an electric field by field lines, Faraday assumed that the mechanical forces arise in the field as a result of tension and mutual repulsion of the field lines; Faraday conceived the lines as being the axes of certain elastic tubes that fill up the space and tend to expand in cross section and contract in length. Modern field theory does not ascribe any more physical reality to these lines than it does to parallels and meridians on a geographic map or to lines of equal heights on a topographic map. However, the field represented by the lines is as real as the height of a mountain. It is thus no wonder that the field configuration enables us to reveal how the forces appear and visualize their direction and intensity. In Fig. 1.1*a* and *b*, the full lines represent field lines, and the dashed lines the traces (lines) of equipotential surfaces. Knowing the pattern of

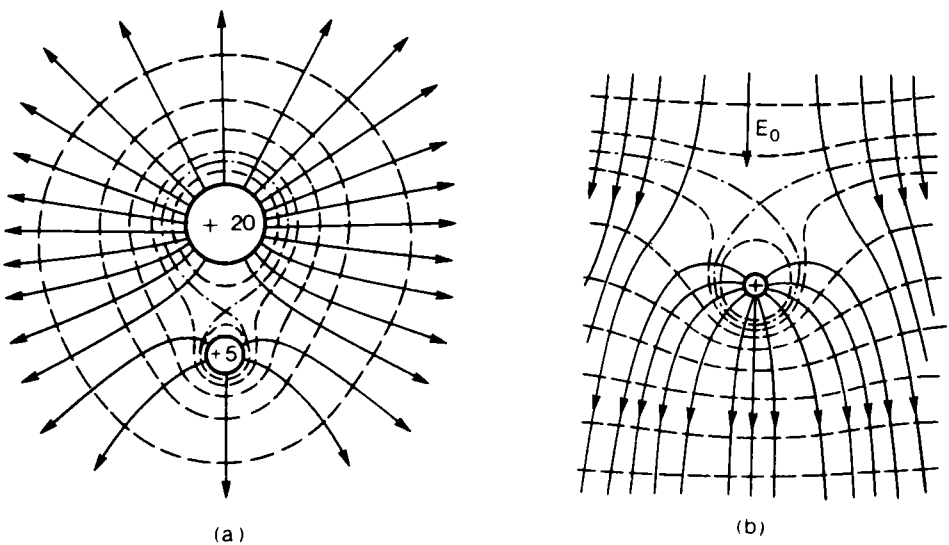


Fig. 1.1

field vectors that surround a body, we can determine the force acting on this body. The figures given here reproduce in a somewhat simplified form the drawings given in Maxwell's treatise. Fig. 1.1*a* illustrates the field of two like charges of which the upper one is four times larger than the lower one; the like charges are seen to repel. Fig. 1.1*b* shows the field of a positive charge introduced into an external uniform field $\mathbf{E}_0$.

1.4. Principal Magnetic Field Vector

Examining the system of two infinitely long thin parallel wires carrying currents i_1 and i_2 (Fig. 1.2*a*), we can express the force acting on a unit length of any of the wires, say, of the second wire, by the formula

$$F_2 = \mu_0(i_1 i_2/2\pi r) \tag{1.14}$$

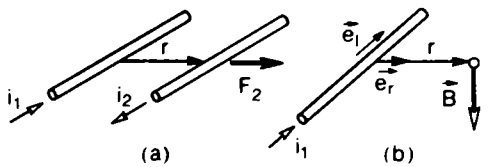


Fig. 1.2

From the viewpoint of field theory, the same result is defined differently. In the space surrounding the moving electric charges, and hence the current, there is a magnetic field. The measure of this field is a vector quantity **B** that determines the flux density. For the historical reasons of development of the magnetic field theory, the quantity **B** is called *magnetic induction*, or *magnetic flux density* (Fig. 1.2*b*).

A moving electric charge in a magnetic field experiences a mechanical force. A stationary charge does not experience any force in a magnetic field. This explains why a magnetic field can be called an electrokinetic field.

If, instead of one charge, we consider a regular motion of charges (Fig. 1.3*a*) characterized by a current density **J**, then the force acting on a volume element dV

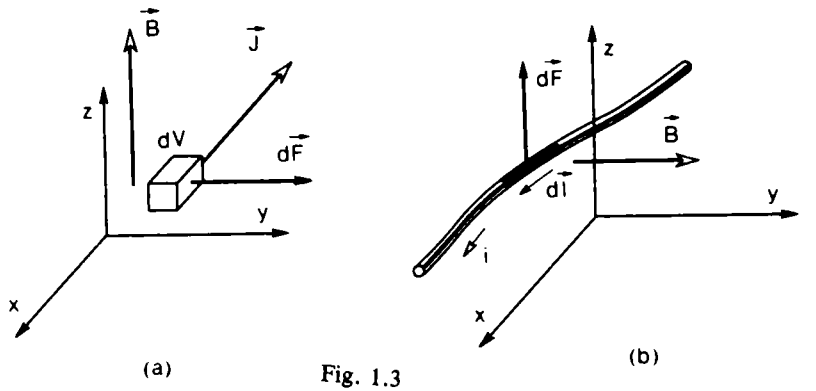


Fig. 1.3

(with a current density **J**) in an external magnetic field **B** will be given by

$$d\mathbf{F} = \mathbf{J} \times \mathbf{B}dV \tag{1.15a}$$

For an element $d\mathbf{l}$ of wire carrying the current i (the $d\mathbf{l}$ is in the direction of current, as shown in Fig. 1.3*b*),

$$d\mathbf{F} = id\mathbf{l} \times \mathbf{B} \tag{1.15b}$$

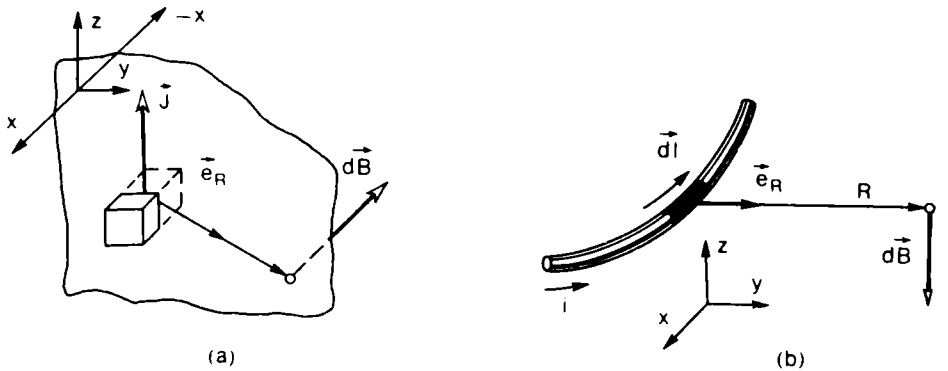


Fig. 1.4

In a uniform field **B**, a straight wire of length l with current i experiences a force

$$\mathbf{F} = i\mathbf{e}_l \times \mathbf{B}l \tag{1.15c}$$

Here $\mathbf{e}_l$ is a unit vector parallel to the wire axis and oriented in a chosen positive direction of current.

Note that in electrical equipment the forces of interaction between the electric field and charges are generally very insignificant; on the contrary, the forces acting on current-carrying conductors in a magnetic field are often very large.

Comparing (1.15c) and (1.14), we may say that in the region of location of the wire with current $i = i_2$, a field of the first (infinitely long) wire exists:

$$\mathbf{B} = \mu_0(i_1/2\pi r)\mathbf{e}_l \times \mathbf{e}_r \tag{1.16}$$

The direction of the field **B** is determined by the vector product of unit vectors $\mathbf{e}_l$ and $\mathbf{e}_r$ (see Fig. 1.2*b*).

Being immersed in this field, an element of the second wire experiences a force given by (1.14).

The magnetic flux density due to a current element is

$$d\mathbf{B} = \mu_0(\mathbf{J} \times \mathbf{e}_R/4\pi R^2)\, dV \tag{1.17a}$$

or

$$d\mathbf{B} = \mu_0(id\mathbf{l} \times \mathbf{e}_R)/4\pi R^2 \tag{1.17b}$$

In these formulas, dV is the elemental volume where the current has a density **J** (Fig. 1.4*a*) and $d\mathbf{l}$ is the elemental length of wire carrying the current i (Fig. 1.4*b*). The total magnetic field vector **B** can be found from formulas (1.17) on integrating over all the wires carrying the current. Formulas (1.16) and (1.17) are valid in the absence of magnetized bodies. They hold for vacuum and, to a high degree of accuracy, for air, copper, and other nonmagnetic substances (see. Ch. 4).

As in the case of an electric field, the forces acting in a magnetic field on current-carrying wires can be evaluated from the pattern of field lines if, following Faraday's conception, we assume that these lines exhibit the properties of elastic tension and contraction. Fig. 1.5 shows pictorially the flux lines, i.e. the lines parallel to the vector **B**, for two parallel wires carrying currents in the same direction (Fig. 1.5*a*) and in

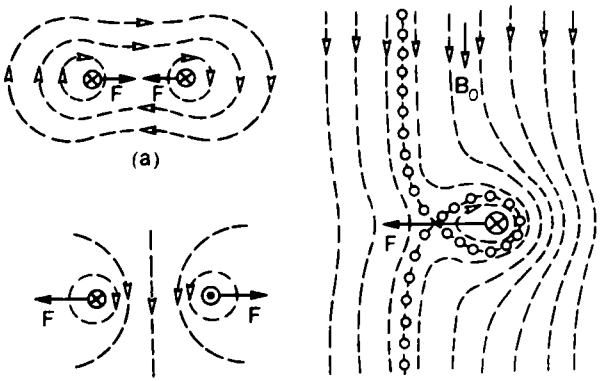


Fig. 1.5

opposite directions (Fig. 1.5*b*), and also the resultant magnetic field (Fig. 1.5*c*) of a current-carrying conductor immersed in an external uniform field $\mathbf{B}_0$. The conductors run normal to the plane of the paper; the cross is the tail and the dot is the head of the arrow indicating the current direction.

1.5. Principle of Superposition

The electric field of a few charges at a given point is the sum of the individual fields of charges, each taken separately. Thus the resultant field of three charges q_1, q_2, and q_3 is

$$\mathbf{E} = \mathbf{E}_1 + \mathbf{E}_2 + \mathbf{E}_3 \tag{1.18}$$

The field $\mathbf{E}_k$ for any kth point charge is determined from formula (1.8) by introducing into it a radius $\mathbf{R}$ drawn from the point of charge location to an observation point.

The above statement expresses the principle of superposition or, saying it in the physical language, the principle of independent action: the fields of charges *1, 2, 3* . . . add up, with the field of each charge remaining the same as it would be found in the absence of the other fields.

If a charge is continuously distributed throughout a volume V at a given charge density ρ, then each element dV contributes its own component of the field at the point of observation in accordance with (1.8):

$$d\mathbf{E} = (\rho dV / 4\pi\varepsilon_0 R^2)\mathbf{e}_R \tag{1.19}$$

As a matter of fact, ρ can be a function of coordinates. The sought-for field is

$$\mathbf{E} = \int (\rho / 4\pi\varepsilon_0 R^2)\mathbf{e}_R dV \tag{1.20}$$

Employing the same principle of superposition for estimating the potential by (1.11) gives a simpler expression since we have a scalar function under the integral:

$$\varphi = \int (\rho / 4\pi\varepsilon_0 R) dV \tag{1.21a}$$

When dealing with point charges q, line charges distributed along the line l with a density τ, or surface charges distributed on the surface S with a density ξ, we should

add the following components to (1.21a):

$$\varphi_q = \Sigma q_i / 4\pi\varepsilon_0 R_i$$
$$\varphi_\tau = \int (\tau / 4\pi\varepsilon_0 R) dl \qquad (1.21b)$$
$$\varphi_\xi = \int (\xi / 4\pi\varepsilon_0 R) dS$$

The values of q_i, τ and ξ can be evaluated by passing to the limit of distribution of a finite volume density ρ. Assume a charge with a volume density ρ distributes itself in a thin surface layer of thickness b. The charge per unit surface area will then be equal to $b\rho$. If $b \to 0$, while $b\rho$ remains finite, we can say that the surface charge has a density

$$\xi = \lim_{b \to 0} b\rho$$

The superposition principle is also applicable to a magnetic field. For a specified distribution of current density, we find from (1.17a) that

$$\mathbf{B} = \mu_0 \int (\mathbf{J} \times \mathbf{e}_R / 4\pi R^2) dV \qquad (1.22a)$$

on integrating over the entire region where $\mathbf{J} \neq 0$. Knowing the geometric location of a wire carrying a line current i, we obtain from (1.17b)

$$\mathbf{B} = -\mu_0 i \int \mathbf{e}_R \times d\mathbf{l} / 4\pi R^2 \qquad (1.22b)$$

on integrating along the entire current loop.

It is sometimes convenient to deal with surface current whose density $\mathbf{k}$ is found as follows. Let a thin layer of thickness b adjacent to the surface have a current of density $\mathbf{J}$, whose vector is certainly parallel to the surface, so that $\mathbf{J}d\mathbf{S} = 0$. If the product $\mathbf{J}b$ remains finite at $b \to 0$, the vector of surface current is equal to

$$\mathbf{k} = \lim_{b \to 0} \mathbf{J}b$$

The magnetic field due to a surface current is

$$\mathbf{B} = \mu_0 \int (\mathbf{k} \times \mathbf{e}_R / 4\pi R^2) dS \qquad (1.22c)$$

The vectors $\mathbf{R} = R\mathbf{e}_R$ in expressions (1.22) also extend from the source points (points of current elements $\mathbf{J}dV$, $id\mathbf{l}$, or $\mathbf{k}dS$) to the observation point (field point).

All the expressions given in this section are valid for nonpolarizable media ($\varepsilon = 1$, $\mu = 1$).

1.6. Field Calculation by Equations in Integral Form

In view of the integral formulation of field equations, it is quite easy to calculate a field, but only in the case of a high degree of symmetry, indicative of constancy of the field vectors on the contour or surface being considered; as a result we can take the sought-for constant quantity outside the integral. Although such cases seem exceptional, they are often met with in practice. The calculations become very simple, more illustrative, and help acquire skill in using the basic field equations. The principle of superposition applied here offers the possibility of enlarging the number of

easily soluble problems. The skill in performing these calculations with clear understanding of the calculation procedure may be regarded to be a prerequisite for better assimilation of the material presented in the subsequent chapters of this book. In this section we give two examples of calculation.

Example 1.1. A sphere of radius a contains an electric charge of constant density ρ. Electrically polarizable bodies are not present ($\varepsilon = 1$). Determine the field and the potential corresponding to this charge at $R \leqslant a$ and $R \geqslant a$.

Solution. Let us surround the charge with a fictitious concentric sphere of radius R and use Gauss' theorem

$$\oint \mathbf{E} d\mathbf{S} = Q/\varepsilon_0$$

From the considerations of symmetry we conclude that $\mathbf{E}$ is equal to $E\mathbf{e}_R$ and constant in magnitude at R = constant. Bearing in mind that $\mathbf{E}d\mathbf{S} = EdS$, and taking the constant quantity E outside the integral sign, we find

$$E \cdot 4\pi R^2 = Q/\varepsilon_0$$

At $R \leqslant a$, the charge residing inside the surface of integration is

$$Q = \rho \cdot 4\pi R^3/3$$

and, hence, in the internal region

$$E = E_i = \rho R/3\varepsilon_0 \tag{a}$$

At $R \geqslant a$, the charge is independent of the radius and equal to the total charge of the sphere:

$$Q = Q_0 = \rho \cdot 4\pi a^3/3$$

and, therefore, in the external region

$$E = E_e = Q_0/4\pi\varepsilon_0 R^2 \tag{b}$$

The direction of the field in both regions is along the radius. The respective potentials are

$$\varphi_i = -\rho R^2/6\varepsilon_0 + K_i \tag{c}$$

$$\varphi_e = Q_0/4\pi\varepsilon_0 R + K_e \tag{d}$$

where K_i and K_e are constants. Assuming the potential at an infinitely remote point to be zero, we set $K_e = 0$. Considering that $\varphi_i = \varphi_e$ at $R = a$, then $K_i = \rho a^2/2\varepsilon_0$. The final expressions are:

$$\varphi_i = (\rho/6\varepsilon_0)(3a^2 - R^2), R \leqslant a \tag{e}$$

$$\varphi_e = Q_0/4\pi\varepsilon_0 R, R \geqslant a \tag{f}$$

The field of an electric dipole. An *electric dipole* is the pair of opposite charges, $+q$ and $-q$, separated by a distance $\mathbf{l}$ called the *moment arm*. The dipole moment is

$$\mathbf{p} = q\mathbf{l} \tag{1.23}$$

At a distance $\mathbf{R} = R\mathbf{e}_R$ from the dipole center, if $R \gg l$, the dipole potential

$$\varphi_p = \mathbf{p}\mathbf{e}_R/4\pi\varepsilon_0 R^2 \tag{1.24}$$

does not depend on either the charges q or the moment arm $\mathbf{l}$, and only depends on

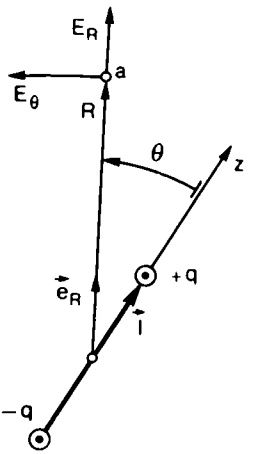


Fig. 1.6

the product of these quantities, i.e. on the dipole moment.

In accordance with the statement of a gradient, the field components in the spherical coordinate system are equal to respective derivatives:

$$E_R = -\partial\varphi/\partial R = p\cos\theta/2\pi\varepsilon_0 R^3 \tag{1.25a}$$

$$E_\theta = -\partial\varphi/R\partial\theta = p\sin\theta/4\pi\varepsilon_0 R^3 \tag{1.25b}$$

Here the origin of spherical coordinates lies in the middle of the arm and the z axis is in the direction of the arm (Fig. 1.6).

Example 1.2. Use the superposition principle to derive formula (1.24), assuming the expression for a point charge potential to be known.

Solution. Let the distance from the dipole center to the observation point a be $\mathbf{R} = R\mathbf{e}_R$. The distance from charges $\pm q$ to the same point a is $\mathbf{R}_{1,2} = \mathbf{R} \mp \mathbf{l}/2$. For a region with $R/l \gg 1$, we find

$$R_{1,2} = \sqrt{R^2 \mp \mathbf{Rl} + l^2/4} = R \mp \mathbf{e}_R\mathbf{l}/2 \tag{a}$$

using formula $\sqrt{1+\varepsilon} = 1 + \varepsilon/2$ at $\varepsilon \ll 1$.

According to the superposition principle, the dipole potential is

$$\varphi = (q/4\pi\varepsilon_0)(1/R_1 - 1/R_2) = q(R_2 - R_1)/(4\pi\varepsilon_0 R_1 R_2) \tag{b}$$

Substituting (a) into (b) and setting $l^2 \ll R^2$ yields (1.24).

Many techniques of calculation of a magnetic field are similar to those used for the calculation of an electric field. Consider the magnetic field of a coaxial cable (Fig. 1.7) in which the current i passing through the conductor comes back along the sheath. The permeability μ of both conducting and insulating materials is equal to unity. The cable may be thought of as being straight and infinitely long; the latter assumption implies that we disregard the field of currents at the points of connection of the cable to the source and load, whatever the configuration of wires at the connection points.

At $\mathbf{B} = \mu_0\mathbf{H}$ and with the condition of symmetry being observed, it follows from

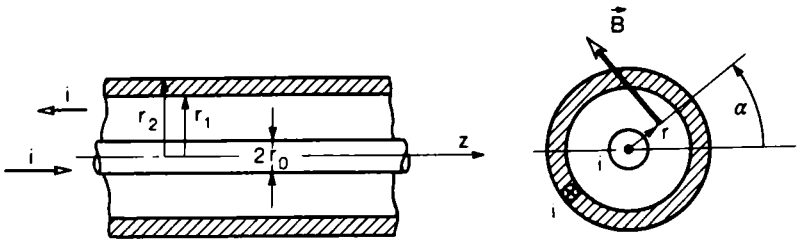


Fig. 1.7

the circuital law (1.1) that inside the conductor ($r \leqslant r_0$)

$$B = B_\alpha = \mu_0 ir/2\pi r_0^2 \tag{1.26}$$

and between the conductor and sheath ($r_0 \leqslant r \leqslant r_1$)

$$B = B_\alpha = \mu_0 i/2\pi r \tag{1.27}$$

and, last, inside the sheath ($r_1 \leqslant r \leqslant r_2$)

$$B = B_\alpha = \mu_0 i(r_2^2 - r^2)/2\pi r(r_2^2 - r_1^2) \tag{1.28}$$

Using formulas (1.22), we can obtain the same results, though the latter approach involves very complicated calculations, particularly for (1.26) and (1.28).

Consider now a long round ferrite core (Fig. 1.8) where a time-varying magnetic field $B = B_z(t)$ exists.

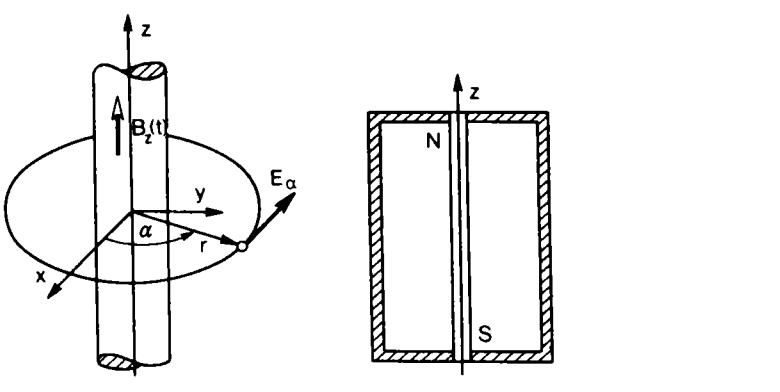
The core radius is r_0; inside the core, the magnetic flux density is independent of coordinates; outside the core, $B = 0$.

From the consideration of symmetry, the electric field caused by variations in the magnetic field has only the αth component and only depends on the radius

$$E = E_\alpha(r) \tag{1.29}$$

From Maxwell's equation (1.2) it then follows that

$$E = -(r/2)(\partial B/\partial t), r \leqslant r_0 \tag{1.30a}$$



(a) (b)

Fig. 1.8

$$E = -(r_0^2/2r)(\partial B/\partial t), r \geqslant r_0 \tag{1.30b}$$

This conclusion presupposes a rather slow-varying field such that we can ignore (1) the process of propagation of electromagnetic waves around the core and also disregard (2) the density of displacement currents ($\mathbf{J}_d = \varepsilon\varepsilon_0 d\mathbf{E}/dt$) and thus the magnetic field which, in turn, accompanies these currents.

Chapter 2 DIFFERENTIAL FIELD EQUATIONS

The laws and relationships reduced to simple form hold for describing various electromagnetic phenomena. In the general theory, however, of most importance are the field equations expressed in differential form. It should be pointed out that students often experience difficulties in understanding these equations if they insufficiently clearly comprehend vector operations of differentiation and are short of experience in their application. Although for proper understanding of the differential equations and their solutions, as presented in this book, it is generally enough to know only a minimum of information on vector analysis, the knowledge of basic operations, symbols, and theorems is of course quite essential.

2.1. Differential Operators[1]

It is easy to transform the integral statements to their differential form by applying vector operations, divergence and rotation. As for the definition of gradient (1.13), see Sec. 1.3.

Divergence of any vector $\mathbf{F}$ is defined as the limit of the quotient of the outward flow of $\mathbf{F}$ through a closed surface S by the volume V within the surface, provided the surface contracts around the given point for which the divergence is determined; the statement certainly implies that the distance from the given point to any point on the surface approaches zero and hence the volume V approaches zero too. The formula for the above statement is

$$\operatorname{div}\mathbf{F} = \lim_{V \to 0} (\oint \mathbf{F} d\mathbf{S})/V \tag{2.1}$$

where div is an adopted abbreviation for divergence.

Rotation or curl of any vector $\mathbf{F}$ is a vector. Its component normal to the surface S is defined as the limit of the quotient of the circulation of $\mathbf{F}$ around the contour l bounding the surface by the surface area S, provided the contour contracts to the given point for which the component of curl is determined; the statement of course implies that the distance from the given point to any point on the contour approaches

[1] See also Appendix A3, formulas (6) to (8).

zero and hence the area S enclosed by the contour approaches zero too. In mathematical form, the statement is given by

$$\operatorname{curl}_n \mathbf{F} = \lim_{S \to 0} \left(\oint \mathbf{F} d\mathbf{l}\right)/S \tag{2.2}$$

where the symbol on the left of the equation stands for the nth component of the curl; the symbol for the vector itself has no subscript: curl $\mathbf{F}$. Notice also that the direction of the normal n to the surface S follows the rule of a right-hand screw turning about its axis and traveling in the direction of the path $d\mathbf{l}$ around the contour bounding the surface S.

The above definitions do not depend on the system of coordinates. It is always possible to apply these definitions to find the vector operations and thus express the divergence and circulation in any system of coordinates (see Appendix A5).

Maxwell's first equation. Substituting (1.1) for total current into the right side of (2.2), we find directly that

$$\operatorname{curl}_n \mathbf{H} = \lim_{S \to 0} \left(\oint \mathbf{H} d\mathbf{l}\right)/S = \lim_{S \to 0} \left(\int \mathbf{J}_t d\mathbf{S}\right)/S \tag{2.3}$$

Applying now the mean-value theorem to the right side of the equation, we get

$$\int \mathbf{J}_t d\mathbf{S} = \bar{J}_{t,n} \int dS = \bar{J}_{t,n} S \tag{2.4}$$

In these formulas, the subscript n denotes the component normal to the surface element $d\mathbf{S}$, and the dash above the letter identifies the mean value equal to the value of the quantity in question, at least at one point on the surface S.

From Eqs. (2.3) and (2.4) it follows that the nth component of the curl of magnetic field strength is equal to the nth component of the vector of current density; we should add that the statement is true for any nth component. This means that we can directly equalize the vectors proper, the curl $\mathbf{H}$ and $\mathbf{J}_t$:

$$\operatorname{curl} \mathbf{H} = \mathbf{J}_t = \mathbf{J} + \partial \mathbf{D}/\partial t \tag{2.5}$$

For an ordinary conducting medium, the density of conduction current can be expressed in terms of electric field strength and conductivity (Ohm's law). Maxwell's first equation then assumes the form

$$\operatorname{curl} \mathbf{H} = \sigma \mathbf{E} + \partial \mathbf{D}/\partial t \tag{2.6}$$

or

$$\operatorname{curl} \mathbf{H} = (\sigma + \varepsilon\varepsilon_0 \partial/\partial t)\mathbf{E} \tag{2.7}$$

if the permittivity ε of the medium is constant in time and independent of vector $\mathbf{E}$.

In case of simple harmonic variations of the vectors under consideration, it is definitely possible to represent these vectors by complex quantities and use the factor $j\omega$ instead of time differentiation. Eqs. (2.6) and (2.7) then convert to phasor expressions

$$\operatorname{curl} \dot{\mathbf{H}} = \sigma \dot{\mathbf{E}} + j\omega \dot{\mathbf{D}} \tag{2.8}$$

and

$$\operatorname{curl} \dot{\mathbf{H}} = (\sigma + j\omega\varepsilon\varepsilon_0)\dot{\mathbf{E}} \tag{2.9}$$

Maxwell's second equation. Employing the same formula (2.2) and substituting Eq. (1.2) into (2.2), we find in a similar manner

$$\operatorname{curl}_n \mathbf{E} = \lim_{S \to 0} [-(\partial/\partial t)(\bar{B}_n \int dS)/S] = -\partial B_n/\partial t \tag{2.10}$$

Since the latter identity is true for any components, it also holds for vectors

$$\operatorname{curl} \mathbf{E} = -\partial \mathbf{B}/\partial t \tag{2.11}$$

For a medium with permeability μ being constant in time and independent of the value of vector, the expression is

$$\operatorname{curl} \mathbf{E} = -\mu\mu_0 \partial \mathbf{H}/\partial t \tag{2.12}$$

For quantities harmonically varying with time, Maxwell's second equation may be written in complex form

$$\operatorname{curl} \dot{\mathbf{E}} = -j\omega \dot{\mathbf{B}} \tag{2.13}$$

and

$$\operatorname{curl} \dot{\mathbf{E}} = -j\omega\mu\mu_0 \dot{\mathbf{H}} \tag{2.14}$$

Gauss' theorem for an electric field. Substituting (1.3) into (2.1) gives the third field equation

$$\operatorname{div} \mathbf{D} = \lim_{V \to 0} (\int \rho dV)/V = \lim_{V \to 0} (\bar{\rho} \int dV)/V = \rho \tag{2.15}$$

Here too, apart from the divergence expression, we have used the mean-value theorem in deriving this equation ($\bar{\rho}$ is the mean value of charge density in volume $V \to 0$).

The derived differential equation says that the divergence of a vector flux density or the density of sources of this vector is equal to the density of an electric charge. This equation is a differential statement of Gauss' theorem for electrostatics.

With the permittivity being constant and also independent of coordinates,

$$\operatorname{div} \mathbf{D} = \operatorname{div} (\varepsilon\varepsilon_0 \mathbf{E}) = \varepsilon\varepsilon_0 \operatorname{div} \mathbf{E} = \rho \tag{2.16}$$

and

$$\operatorname{div} \mathbf{E} = \rho/\varepsilon\varepsilon_0 \tag{2.17}$$

Laplace-Poisson's equation. In cases where an electric field is determined by charges described by Coulomb's law (1.8), (1.11) and (1.20), (1.21), its strength may be expressed as a potential gradient

$$\mathbf{E} = -\operatorname{grad} \varphi \tag{2.18}$$

Substituting the latter expression in (2.17) gives

$$\operatorname{div} \operatorname{grad} \varphi = -\nabla^2 \varphi = -\Delta\varphi = \rho/\varepsilon\varepsilon_0 \tag{2.19}$$

This formula represents Poisson's equation when $\rho \neq 0$, or Laplace's equation when $\rho = 0$.

In formula (2.19), ∇^2 is the del (nabla) operator (nabla square) and Δ is the Laplacian operator or the laplacian; when this operator of the second order is applied to a scalar, for example, to φ, the above operators are identically equal:

$$\operatorname{div} \operatorname{grad} \equiv \nabla^2 \equiv \Delta \tag{2.20}$$

Laplace-Poisson's equation is often written as

$$\Delta\varphi = \begin{cases} 0 & \text{Laplace's equation} \quad (2.21a) \\ -\rho/\varepsilon\varepsilon_0 & \text{Poisson's equation} \quad (2.21b) \end{cases}$$

Gauss' theorem for a magnetic field. Applying the definition of divergence (2.1) to (1.4) in integral form, we can convert the latter to a differential expression

$$\operatorname{div}\mathbf{B} = 0 \tag{2.22}$$

which implies that the magnetic **B** field vector has no source. This is equivalent to such a formulation in accordance with Faraday's law: the magnetic lines of field **B** always form closed loops, i.e. they terminate on nothing but themselves.

In analogy with the divergence of electric flux density (2.16), the right side of (2.22) should be expected to equal the density of a magnetic charge. But it so happens that no magnetic charge exists, at least, it has not been found to be present in matter.

Relations between vectors. Maxwell's first two equations contain five vectors. Obviously, there must be three more equations that independently establish the relations between these vectors. The first of these relations is Ohm's law given in the set of three equations (1.5), of which the other two can represent the remaining vector relations.

Notes: 1. **Relating to continuity and differentiability.** Integral expressions (2.1) and (2.2) for the limiting cases do not require further stipulations only when the integrands of coordinates are continuous and differentiable; the same concerns the mean-value theorem.

Discontinuity appears if point, line, or surface charges or currents are present [see Eqs. (1.21b), (1.22b), and (1.22c)]. It is sometimes possible to avoid special formulations by considering the distribution of charges and currents of finite density in a certain finite volume which then approaches zero. In certain cases the concept of discontinuous functions expressing charge distribution proves useful and sometimes necessary. Even such basic formulas as Laplace's and Poisson's equations are directly applicable only for the regions of continuous charge distribution (see example 2.2).

2. **The del symbol in designating vector operations.** The gradient, divergence, and curl can always be expressed in terms of the vector operator del:

$$\nabla\varphi \equiv \operatorname{grad}\varphi, \quad \nabla\mathbf{B} \equiv \operatorname{div}\mathbf{B}$$

$$\operatorname{curl}\mathbf{E} \equiv \nabla \cdot \mathbf{E}, \quad \nabla^2\varphi \equiv \operatorname{div}\operatorname{grad}\varphi$$

where the sign $\cdot$ denotes the vector product.

In cartesian (rectangular) coordinates,

$$\nabla \equiv \mathbf{e}_x(\partial/\partial x) + \mathbf{e}_y(\partial/\partial y) + \mathbf{e}_z(\partial/\partial z)$$

The list of vector operations considered here appears in appendix A5 for the cartesian, cylindrical, and spherical orthogonal system of coordinates.

2.2. Examples of Operations on Various Fields

Consider simple examples of fields illustrated schematically in figures presented below.

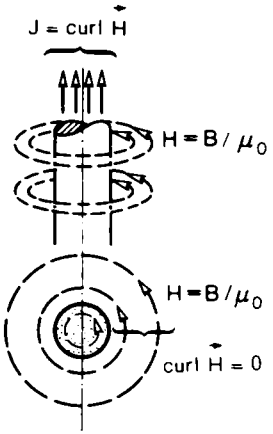


Fig. 2.1

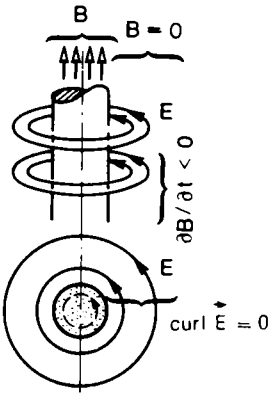


Fig. 2.2

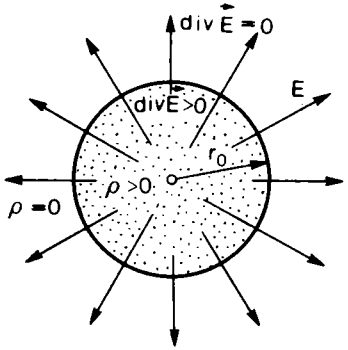


Fig. 2.3

Figure 2.1 shows a current-carrying conductor with a magnetic field depicted as a number of coaxial loops. The field lines are seen to be source-free (div $\mathbf{H} = 0$). In the internal region of the conductor, curl $\mathbf{H} = \mathbf{J} \neq 0$, while in the external region where no current exists, curl $\mathbf{H} = 0$. In both regions, div $\mathbf{H} = 0$. We can readily ascertain the results by carrying out the appropriate operation on (1.26) and (1.27).

Figure 2.2 illustrates the electric field due to variations in magnetic induction inside a long rod. The electric field configuration is similar to that of Fig. 2.1. In accordance with the second field equation

$$\operatorname{curl} \mathbf{E} = -\partial \mathbf{B}/\partial t$$

and therefore in the internal region, curl $\mathbf{E}$ is parallel to the vector $\mathbf{B}$ when $B = B_z > 0$ and $\partial B/\partial t < 0$; otherwise, the curl $\mathbf{E}$ would be antiparallel to the vector $\mathbf{B}$. It is easy to check the statements by appropriately differentiating expressions (1.30).

Figure 2.3 shows the field of a space charge distributed with a density $\rho > 0$ inside the cylinder of radius r_0; outside the cylinder, $\rho = 0$. As seen, the origins of field lines leaving out the circle lie within the region $r < r_0$, where div $\mathbf{E} > 0$. Outside the region $r > r_0$, the lines extend continuously and will certainly end on negative charges in space.

If the region $r < r_0$ had a space charge formed by electrons, the arrows of vectors representing the $\mathbf{E}$ field would change direction and therefore ρ and div $\mathbf{E}$ would change sign.

Example 2.1. Show that expressions (a), (b), (e), and (f) of Example 1.1 correspond to the constant charge density ρ inside the sphere and to the absence of charge outside the sphere.

Solution. Calculating the divergence of the electric field and the laplacian of the potential, we obtain the required result. Operations should be performed in spherical coordinates (see Appendix A5).

Example 2.2. The potential is expressed as

$$\varphi = (b/4\pi\varepsilon_0 R)e^{-aR} \tag{a}$$

where R is the radius of the spherical coordinate system; the remaining quantities are constants.

Verify now whether Gauss' theorem

$$Q = \int \rho dV = \varepsilon_0 \oint \mathbf{E} d\mathbf{S} \tag{b}$$

is applicable to the field in question. Also, determine whether Poisson's equation

$$-\nabla^2 \varphi = \rho / \varepsilon_0 \tag{c}$$

holds for evaluating the charge density. Besides, find whether it is possible to calculate the charge in volume V by integrating the charge density over the volume. Explain the results obtained.

Solution. Calculate first the field strength. In the case when the potential is the function of the radius alone,

$$\operatorname{grad} \varphi = \mathbf{e}_R \partial \varphi / \partial R \tag{d}$$

In our case,

$$\mathbf{E} = -\operatorname{grad} \varphi = (b/4\pi\varepsilon_0 R^2)(1 + aR)e^{-aR}\mathbf{e}_R \tag{e}$$

The laplacian of φ, here too, when $\varphi = \varphi(R)$,

$$\nabla^2 \varphi \equiv \Delta \varphi = (1/R)[\partial^2 (R\varphi)/\partial R^2] = (ba^2/4\pi\varepsilon_0 R)e^{-aR} \tag{f}$$

and, hence, by Poisson's equation,

$$\rho = -\varepsilon_0 \nabla^2 \varphi = -(ba^2/4\pi R)e^{-aR} \tag{g}$$

Substituting (g) into the middle term of equality (b) yields

$$\begin{aligned} \int \rho dV &= -ba^2/4\pi \int_0^R (e^{-aR}/R) 4\pi R^2 dR \\ &= -b \int_0^R (aR) e^{-aR} d(aR) = b[(1 + aR)e^{-aR} - 1] \end{aligned} \tag{h}$$

Substituting (e) into the right side of (b) gives another result:

$$\varepsilon_0 \oint \mathbf{E} d\mathbf{S} = \varepsilon_0 E_R \cdot 4\pi R^2 = b(1 + aR)e^{-aR} \tag{i}$$

The obvious difference in the results is inconsistent with equality (b). The integral (h), i.e. $\int \rho dV$, is smaller than the corresponding surface integral (i), i.e. $\varepsilon_0 \oint \mathbf{E} d\mathbf{S}$, by a value of b irrespective of R. From this we can assume that on integrating the charge over the volume, the point charge b at the origin has not been taken into account. The presence of this charge corresponds to a particular field point at which Poisson's equation is inapplicable.

A more general expression for the charge inside the volume

$$Q = \Sigma_V q + \int_V \rho dV \tag{j}$$

(the assumption being the same as above) gives

$$Q = b(1 + aR)e^{-aR} \tag{k}$$

In this case Gauss's theorem of electrostatics

$$Q = \varepsilon_0 \oint \mathbf{E} d\mathbf{S}$$

is certainly true.

2.3. Displacement Current in Vacuum

Consider a field in the absence of polarizable dielectrics (when $\mathbf{P} = 0$ and $\varepsilon = 1$) and polarizable ferromagnets (when $\mathbf{M} = 0$ and $\mu = 1$). These conditions are true for vacuum and, to a certain approximation, for air.

In this case Maxwell's first equation can be written in the form

$$\operatorname{curl} \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \varepsilon_0 \partial \mathbf{E}/\partial t \tag{2.23}$$

In the right side of the equation, the vector $\varepsilon_0 \partial \mathbf{E}/\partial t$ is as important as the vector of current density $\mathbf{J}$ due to the flow of charges.

This means that a change of electric field produces a magnetic field as effectively as the electric current (flow of charges) does. This remarkable result of Maxwell's electrodynamics directly follows from the analysis of the first equation.

For a direct current field ($\partial \mathbf{E}/\partial t = 0$), Maxwell's first equation assumes a simpler form

$$\operatorname{curl} \mathbf{B} = \mu_0 \mathbf{J} \tag{2.24}$$

or

$$\oint \mathbf{B} d\mathbf{l} = \mu_0 \int \mathbf{J} d\mathbf{S} \tag{2.25}$$

where $\mathbf{J} = <\rho \mathbf{v}> = <\rho + \mathbf{v}> + <\rho - \mathbf{v}>$ is the current density caused by a steady transport of charges. The brackets $< >$ denote averaging.

The last two equations become meaningless for a varying electric field.

Indeed, the current leaving out any closed surface must be equal to a decrease in the amount of electricity, Q, enclosed within this surface:

$$i = \oint \mathbf{J} d\mathbf{S} = -\partial Q/\partial t = -(\partial/\partial t) \int \rho dV \tag{2.26}$$

The expression is the statement of the principle of conservation of charge.

Replacing the right side of the last equality by (2.16) and using the Ostrogradsky-Gauss divergence theorem (see formula 2 in Appendix A3), we find that

$$\oint \mathbf{J} d\mathbf{S} = -(\partial/\partial t) \oint \varepsilon_0 \mathbf{E} d\mathbf{S} = -\oint \varepsilon_0 (\partial \mathbf{E}/\partial t) d\mathbf{S} \tag{2.27}$$

This expression implies that the current density vector flux through the bounding surface may not be equal to zero when there is a varying field ($\partial \mathbf{E}/\partial t \neq 0$).

In this case the notion itself — the vector flux of current density $\mathbf{J}$ across the surface bounded by the contour l (2.25) — has no sense. The concept of a flux through a contour, or more accurately a flux across the surface bounded by the closed loop, acquires a definite meaning only if the same flux passes through any surface bounded by the contour. But the latter condition implies that the vector flux across the closed surface is equal to zero. Expression (2.25) is thus invalid in the presence of a varying electric field ($\partial \mathbf{E}/\partial t \neq 0$), as is obvious from (2.27).

Let us use Eq. (2.25) for a circuit completed through a capacitor (Fig. 2.4). An alternating current can flow in such a circuit, and even a direct current can do as well for a short time if the voltage

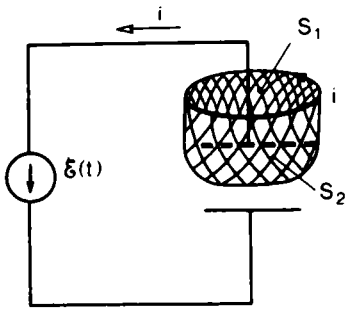


Fig. 2.4

across the capacitor varies with time. The direct current flows in the circuit if the voltage across the capacitor linearly rises with time:

$$i = Cdu_C/dt = \text{constant at } u_C = U_0 t/T$$

Draw a closed loop l so that it encircles one of the wires. For this loop, $\oint \mathbf{B}d\mathbf{l}$ is determined unambiguously. Draw then a surface S_1 (see Fig. 2.4) within the loop so that it will traverse the wire carrying a current i. In this case, the right side of (2.25) is equal to $\mu_0 i$. Deform now the surface of integration within the same loop in such a manner that the current-carrying conductor cannot intersect it. In Fig. 2.4, this is the surface S_2 that passes through the capacitor and thus does not intercept the current i. For this surface, the right side of (2.25) is equal to zero.

The analysis of Maxwell's first equation (2.24) written in differential form also clearly evidences that an additional term is necessary.

Really, assuming that the integrals in (2.26) are taken over the volume V and surface S bounding this volume, we divide these integrals by V and, after passing to the limit, find that

$$\lim_{V \to 0} (\oint \mathbf{J}d\mathbf{S})/V = \lim_{V \to 0} [-\int (\partial\rho/\partial t)dV]/V \qquad (2.28)$$

By applying the mean-value theorem to the right side of the equality and noting that the left side is nothing else than the statement of divergence, we arrive at the law of conservation of electricity in differential form:

$$\operatorname{div} \mathbf{J} = -\partial\rho/\partial t \qquad (2.29)$$

But this statement is at variance with (2.24) since *the divergence of any curl is identically equal to zero* in the sense of mathematical operations proper. Taking the divergence of both sides of (2.24) and considering (2.29), we obtain

$$\operatorname{div} \operatorname{curl} \mathbf{B} \equiv 0 \neq \mu_0 \operatorname{div} \mathbf{J} = -\mu_0 \partial\rho/\partial t \qquad (2.30)$$

Consequently, in order that (2.24) might be valid for varying fields too, its right

side should contain a term **F** which reduces to zero the divergence of the right-side term:

$$(1/\mu_0)\,\mathrm{curl}\,\mathbf{B} = \mathbf{J} + \mathbf{F}$$

at

$$\mathrm{div}\,(\mathbf{J} + \mathbf{F}) \equiv 0 \tag{2.31}$$

Such a vector **F** is easy to find. Expressing the charge density in (2.29) by Gauss' theorem (2.16), we obtain

$$\mathrm{div}\,\mathbf{J} = -\mathrm{div}(\varepsilon_0 \partial \mathbf{E}/\partial t) \tag{2.32}$$

or

$$\mathrm{div}(\mathbf{J} + \varepsilon_0 \partial \mathbf{E}/\partial t) \equiv 0 \tag{2.33}$$

The sought-for term thus assumes the form

$$\mathbf{F} = \varepsilon_0 \partial \mathbf{E}/\partial t \tag{2.34}$$

In the sense of Maxwell, this quantity may be called the density of vacuum displacement current, and the sum

$$\mathbf{J}_t = \mathbf{J} + \varepsilon_0 \partial \mathbf{E}/\partial t \tag{2.35}$$

the density of total (or actual) current.

The divergence of the vector of total current density is identically zero

$$\mathrm{div}\mathbf{J}_t \equiv 0 \tag{2.36}$$

as seen from (2.33). The vector flux of total current density across any closed surface is then zero too:

$$\oint \mathbf{J}_t d\mathbf{S} \equiv 0 \tag{2.37}$$

So, the replacement of current density by total current density in (2.24) and (2.25) makes these equations unambiguous and consistent in any conditions. Thus in the system of Fig. 2.4, the vacuum displacement current through the surface S_2 is equal to the current i passing through the surface S_1.

The introduction of total current into Maxwell's first equation

$$(1/\mu_0)\mathrm{curl}\,\mathbf{B} = \mathbf{J} + \varepsilon_0 \partial \mathbf{E}/\partial t \tag{2.38a}$$

or

$$(1/\mu_0)\oint \mathbf{B} d\mathbf{l} = \int (\mathbf{J} + \varepsilon_0 \partial \mathbf{E}/\partial t) d\mathbf{S} \tag{2.38b}$$

constitutes one of the most important statements of Maxwell's electrodynamic theory. What inevitably stems from this statement is that there must exist quite new phenomena which were earlier difficult or even impossible to imagine. The validity of the above equations was confirmed by experiments.

2.4. Corollaries to Maxwell's Equations

From Eqs. (2.38) it primarily follows that a time-varying electric field gives rise to a magnetic field as does an electric current. In other words, in the absence of current in its common sense, i.e. in the absence of the motion of charges, a magnetic field

must exist, the curl of which is equal to the rate of change of the electric field. But does the assumption of this relation between magnetic and electric fields agree with reality? We can judge of this, first, from a direct experiment, namely, by measuring a magnetic field arising from changes of the electric field, and, second, from the experimental observation of electromagnetic phenomena which must exist as a consequence of the said assumption of the magnetic field due to displacement currents.

It is interesting to note that Maxwell's brilliant hypothesis of the possible existence of a magnetic field without currents and magnetized bodies and also an electric field without electric charges in accordance with the equations

$$\operatorname{curl} \mathbf{B} = \mu_0 \varepsilon_0 \partial \mathbf{E}/\partial t$$

$$\operatorname{curl} \mathbf{E} = -\partial \mathbf{B}/\partial t \tag{2.39}$$

or

$$\operatorname{curl} \dot{\mathbf{B}} = j\omega \mu_0 \varepsilon_0 \dot{\mathbf{E}}$$

$$\operatorname{curl} \dot{\mathbf{E}} = -j\omega \dot{\mathbf{B}}$$

raised doubts and criticism at the time of publication of Maxwell's theory. It seemed to some of his contemporaries that Maxwell reduced real physical phenomena to mathematical abstractions which led to the prediction of existence of electrical phenomena without electricity and magnetic phenomena without current and magnetism (magnetized bodies or permanent magnets). However, what makes Maxwell's theory so important is that he, following Faraday, was advancing the idea of physical reality of an electromagnetic field.

Electromagnetic waves; their propagation with light velocity. Proceeding from Eqs. (2.39), Maxwell clearly showed the possibility of existence of electromagnetic waves propagating with the velocity of light.

Figure 2.5 reproduces Fig. 73 from Sec. 791 of Maxwell's treatise; it illustrates an electromagnetic plane wave moving in the z direction. In Maxwell's terminology, electric displacement for vacuum is $\varepsilon_0 \mathbf{E}$ and magnetic force for vacuum is $\mathbf{B}/\mu_0$; the plus sign indicates that the respective directions are taken positive; the coordinates of the cartesian system shown at the top are added here to the original figure.

The field of this wave traveling with a speed $v = 1/\sqrt{\varepsilon_0 \mu_0} = c$ may be represented by the following expressions:

$$E = E_x = E_0 \sin \omega(t - z/v)$$
$$B = B_y = B_0 \sin \omega(t - z/v) \tag{2.40a}$$

These formulas may also be written as

$$E = E_x = E_0 \sin (\omega t - kz)$$
$$B = B_y = B_0 \sin (\omega t - kz) \tag{2.40b}$$

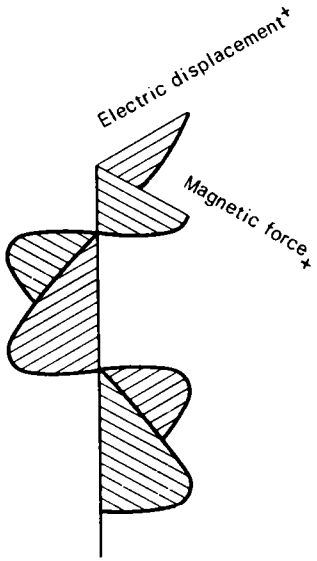


Fig. 2.5

leaving thus ω inside the parantheses. It is this form of writing that is convenient in passing to complex quantities:

$$\dot{E} = \dot{E}_0 e^{-jkz}, \dot{B} = \dot{B}_0 e^{-jkz} \tag{2.40c}$$

Here $\dot{E}$ and $\dot{B}$ are complex amplitude quantities; the real values of these quantities are equal to their moduli divided by $\sqrt{2}$. It is obvious that in the last formulas

$$k = \omega/v \tag{2.41}$$

A wavelength defined by Eqs. (2.40) is easy to find as a z coordinate increment in which the argument varies by 2π. The wavelength is thus equal to

$$\lambda = 2\pi/k = v/f = vT \tag{2.42}$$

It can readily be verified that expressions (2.40) agree with Eqs. (2.39).

Performing operations of differentiation in cartesian coordinates and considering that the field vectors of a plane wave depend on time and only on one spatial coordinate z, we find that

$$\text{curl } \mathbf{B} = \mathbf{e}_x(\omega/v)B_0 \cos \omega(t - z/v)$$

and

$$\mu_0\varepsilon_0 \partial\mathbf{E}/\partial t = \mathbf{e}_x \mu_0\varepsilon_0 \omega E_0 \cos \omega(t - z/v)$$
$$\text{curl } \mathbf{E} = -\mathbf{e}_y(\omega/v)E_0 \cos \omega(t - z/v)$$

and

$$\partial\mathbf{B}/\partial t = \mathbf{e}_y \omega B_0 \cos \omega(t - z/v)$$

Equations (2.39) are satisfied if

$$(\omega/v)B_0 = \mu_0\varepsilon_0\omega E_0 \text{ and } (\omega/v)E_0 = \omega B_0$$

i.e. only if

$$E_0/B_0 = v = 1/\sqrt{\varepsilon_0\mu_0} = \omega/k \tag{2.43}$$

The striking result of this solution lies in that the velocity v proves equal to the velocity of light in all the systems of units and with any method of determining the electric and magnetic constants, ε_0 and μ_0. It is really this result led Maxwell to conclusion that light is an electromagnetic wave. Maxwell even calculated the values of the electric and magnetic vectors of sunlight considering its energy as the energy of a field. He also calculated the value of light pressure.

But Maxwell did not prove directly the validity of his equations. He did not arrive at the conclusion that there exists radiation of electromagnetic waves at electric oscillations; this fascinating effect lies at the root of radio engineering and directly follows from Maxwell's equations. It is an interesting fact that this conclusion does not involve at all complex computations, and Maxwell himself was a splendid master of the required body of mathematics.

Experimental proofs of the validity of Maxwell's equations. Only 15 years after the publication of Maxwell's treatise did Hertz confirm beyond any question the validity of Maxwell's equations. In conducting his experiments, Hertz strived to produce oscillations of a maximum possible frequency in order to increase the derivative of an electric field strength that determines the density of a displacement current. In these experiments Hertz discovered electromagnetic waves emanating from the vibrator (see Fig. 10.2). The discovery of radiation is the basic result of his experiments. Hertz proved that this radiation occurs in conformity with Maxwell's equations. The present book deals with radiation in Ch. 10.

Microwave cavity resonators. High-frequency generators and measuring devices available today make it easy to conduct experiments which clearly and directly attest to the "creation" of a magnetic field by displacement currents in vacuum. Fig. 2.6 illustrates the sectional view of a cylindrical cavity resonator in which a varying electromagnetic field is excited (known as the field of type H_{011}). Dashed lines denote

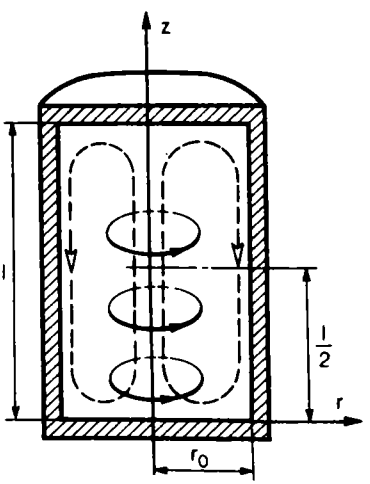


Fig. 2.6

the lines of a magnetic field and solid lines identify the lines of an electric field. The resonator walls are made from a good conductor so that the field does not practically penetrate the walls.

In its configuration, the magnetic field here resembles the magnetic field of a solenoid where the displacement currents closing on themselves in the form of loops serve as a "winding".

Cavity resonators of simpler types are considered in detail in Ch. 9.

2.5. Scalar and Vector Potentials

Scalar potential. It is well known that the vector $\mathbf{E} = \mathbf{E}(x, y, z)$ of any field can be represented by the gradient of a certain scalar function of coordinates, $\varphi = \varphi(x, y, z)$, commonly written with the minus sign,

$$\mathbf{E} = -\operatorname{grad} \varphi \tag{2.44}$$

if and only if this field has a zero curl in the region under observation:

$$\operatorname{curl} \mathbf{E} = 0 \tag{2.45}$$

The latter identity is quite obligatory since mathematical operations imply that the curl of any gradient is identically equal to zero

$$\operatorname{curl} \operatorname{grad} \varphi \equiv \nabla \times \nabla \varphi \equiv 0$$

The scalar function φ is called a *scalar potential.*

Potential field. If in the field being observed curls are nonexistent at all, the field is simply said to be potential without any stipulations. This is exactly the field of stationary or slowly moving charges; it can always be represented as the sum of fields of individual charges (1.18) to (1.20). The field of each charge (in vacuum) according to (1.8) is

$$\mathbf{E} = q\mathbf{e}_R/4\pi\varepsilon_0 R^2$$

The curl of such a field is identically zero. The vector integral taken about a closed path is also identically equal to zero:

$$\oint \mathbf{E} d\mathbf{l} = 0 \tag{2.46}$$

In such a field the potential φ is evaluated uniquely, accurate to an arbitrary additive constant, as follows from the integral

$$\varphi = -\int \mathbf{E} d\mathbf{l} = \int \operatorname{grad} \varphi \, d\mathbf{l} \tag{2.47}$$

If the potential is determined for a point x, y, z, then the integral is bounded only by an upper limit. This does not make the integral definite, and so any arbitrary constant can be added to it.

A potential or, more precisely, a potential difference may by given a definite physical interpretation: the difference in electric potentials of two points a and b is equal to the work per unit charge done by field forces in transporting the charge q. Denoting work by W, the statement can be written in the form

$$(1/q)W_{ab} = (1/q)\int_a^b \mathbf{F} d\mathbf{l} = (1/q)\int_a^b q\mathbf{E} d\mathbf{l} = \int_a^b \mathbf{E} d\mathbf{l} = \varphi_a - \varphi_b \tag{2.48}$$

Field with nonzero curl. In that region of space where curl $\mathbf{F} \neq 0$, the vector $\mathbf{F}$ cannot be represented by the gradient of a scalar potential.

For a coaxial cable (see Fig. 1.7), at $r < r_0$ (i.e. inside the conductor), the magnetic field is

$$B = B_\alpha = \mu_0 ir/2\pi r_0^2 \tag{2.49}$$

This is a rotational field:

$$\operatorname{curl} \mathbf{B} = \mu_0 \mathbf{J} \tag{2.50}$$

It cannot be represented by the gradient of any scalar function.

Vector potential. There is one more possibility of representing the vectors of field $\mathbf{V}$ in terms of a spatial derivative. This representation is possible when

$$\operatorname{div} \mathbf{V} = 0 \tag{2.51}$$

for a vector, i.e. when sources in the vector field do not exist. In this case the vector $\mathbf{V}$ can be represented by the curl of a certain other vector $\mathbf{A}$:

$$\mathbf{V} = \operatorname{curl} \mathbf{A} \tag{2.52}$$

This vector $\mathbf{A}$ is called a *vector potential* of field $\mathbf{V}$ in analogy with the scalar potential whose gradient determines the field being examined. What makes identity (2.51) necessary is that according to the rules of vector operations, the divergence of any curl is identically equal to zero

$$\operatorname{div} \operatorname{curl} \mathbf{A} \equiv \nabla(\nabla \times \mathbf{A}) \equiv 0 \tag{2.53}$$

But a magnetic field exactly satisfies the condition (2.51). The vector of magnetic induction has no sources:

$$\operatorname{div} \mathbf{B} = 0 \tag{2.54}$$

For this reason the vector $\mathbf{B}$ can be specified as the curl of a magnetic vector potential

$$\mathbf{B} = \operatorname{curl} \mathbf{A} \tag{2.55}$$

The magnetic vector potential $\mathbf{A}$ is a function whose curl is equal to the vector $\mathbf{B}$. By Faraday's statement the vector $\mathbf{A}$ serves as a measure of an electrotonic condition. He ascribed a substantial physical meaning to this vector.

It follows from the definition that any function $\mathbf{F}$ whose curl is equal to zero can be added to the vector $\mathbf{A}$, since in this case

$$\operatorname{curl} \mathbf{A} = \operatorname{curl}(\mathbf{A} + \mathbf{F}) \tag{2.56}$$

The arbitrary component $\mathbf{F}$ added to the vector can be any function of coordinates. It is given as the gradient of a scalar

$$\mathbf{F} = \operatorname{grad} \eta \tag{2.57}$$

because the curl of the gradient is identically equal to zero.

The divergence of a curl is identically equal to zero, div curl $\mathbf{S} \equiv \equiv 0$, whereas the divergence of a vector specified as a gradient may be different from zero. Moreover, we can always choose the function $\mathbf{F} = = \operatorname{grad} \eta$ so that div $\mathbf{F}$ is the desired function of coordinates. This

implies that by adding the term $\mathbf{F} = \text{grad}\ \eta$, we can change the divergence of a vector potential, div $\mathbf{A}$, without affecting its curl, curl $\mathbf{A}$. This remarkable property of vector functions is used to simplify equations containing vector potentials. Substituting (2.55) in Maxwell's first equation yields

$$\text{curl}\,\mathbf{B} = \text{curl curl}\,\mathbf{A} = -\nabla^2\mathbf{A} + \text{grad div}\,\mathbf{A} = \mu_0\mathbf{J} \tag{2.58}$$

Based on what has just been said above, we can add any value suitable for us to the term grad div $\mathbf{A}$ by choosing the proper form of function $\mathbf{F}$.

In calculating the fields slowly varying with time, it is common to assume

$$\text{div}\,\mathbf{A} = 0 \tag{2.59}$$

In the general case, it is most suitable to take

$$\text{div}\,\mathbf{A} = -\varepsilon_0\mu_0\partial\varphi/\partial t \tag{2.60}$$

where φ is the scalar potential of an electric field.

When the potential changes but slowly, we may assume that the latter expression for divergence of $\mathbf{A}$ converts to the previous one. This leads to the equation relating the vector potential to current density

$$\nabla^2\mathbf{A} = -\mu_0\mathbf{J} \tag{2.61}$$

This equality permits expressing the vector potential for a dc field as an integral:

$$\mathbf{A} = \mu_0\int(\mathbf{J}dV/4\pi R) \tag{2.62a}$$

or

$$\mathbf{A} = \mu_0\int(i\mathbf{d}\mathbf{l}/4\pi R) \tag{2.62b}$$

Indeed, expression (2.61) is analogous to Poisson's equation (2.19) for an electrostatic field:

$$\nabla^2\varphi = -\rho/\varepsilon_0$$

whose solution is given by (1.21a):

$$\varphi = (1/\varepsilon_0)\int(\rho dV/4\pi R)$$

Rewriting (2.61) for the components in the cartesian coordinate system

$$\nabla^2 A_x = -\mu_0 J_x,\ \nabla^2 A_y = -\mu_0 J_y,\ \ldots$$

we can derive a similar integral expression on the analogy of Poisson's equation. Using it for all components, summing them up after the requisite multiplication by $\mathbf{e}_x$, $\mathbf{e}_y$, and $\mathbf{e}_z$, we obtain (2.62a).

Formula (2.62a) is particularly suitable where $\mathbf{J}$ has the only component in the cartesian system, for example, $J = J_z$.

Without going to further generalization, let us determine the vector potential for particular cases.

Inside a coaxial cable conductor ($r \leqslant r_0$) whose field is described by (1.26), we can readily find that

$$A = A_z = -\mu_0 i r^2/4\pi r_0^2 \tag{2.63}$$

considering the general expression for curl. In cylindrical coordinates,

$$\operatorname{curl} \mathbf{A} = -\mathbf{e}_\alpha(\partial/\partial r)A_z + \mathbf{e}_z(1/r)(\partial/\partial r)(rA_\alpha) \tag{2.64}$$

since $\partial/\partial\alpha = \partial/\partial z = 0$ in accordance with the conditions of the problem.

Besides, it is known that $\mathbf{B} = \operatorname{curl} \mathbf{A}$ has only the αth component. Therefore, we may set $A_\alpha = 0$ and find A_z from the equation

$$B = B_\alpha = \mu_0 i r/2\pi r_0^2 = -dA_z/dr \tag{2.65}$$

The solution to this equation is given by (2.63), with the constant of integration being so chosen that $A = 0$ at $r = 0$.

For the region $r_1 > r > r_0$, where the quantity A_z is determined from the equation

$$B = B_\alpha = \mu_0 i/2\pi r = -dA_z/dr \tag{2.66}$$

the solution is obvious:

$$A_z = -(\mu_0 i/2\pi)\ln r + \text{constant} \tag{2.67}$$

Let us choose an integration constant such that (2.63) and (2.67) give the same result at $r = r_0$ (see note 2 at the end of this section). The constant that meets this requirement is

$$\text{constant} = (\mu_0 i/4\pi)(2 \ln r_0 - 1)$$

Expression (2.67) then assumes the form

$$A_z = -(\mu_0 i/4\pi)[1 + 2\ln(r/r_0)] \tag{2.68}$$

We invite the reader to determine $A_z(r)$ for the region $r \geqslant r_1$.

Field vectors may be expressed as a curl of the vector potential, and also in the region where the curl is zero. This is obvious, for example, from solution (2.68) given above.

The notion of a vector potential is applicable not only to electromagnetic fields but also to other entities. It is possible, and in a number of cases advantageous, to introduce the notion of a vector potential of speed, $\mathbf{v}$, specifying the speed by the equation $\mathbf{v} = \operatorname{curl} \mathbf{A}$; for a cylinder rotating at an angular velocity ω, one may set $A = A_z = -\omega r^2/2$.

Vector potential and the field of a magnetic dipole. Consider a rectangular current loop with sides $\mathbf{l}_1 = -\mathbf{l}_3$ and $\mathbf{l}_2 = -\mathbf{l}_4$. Assume $R = |\mathbf{R}|$ is the distance from the center of the area bounded by the loop to the observation point a (Fig. 2.7). Also, set $R \gg l_k$, where l_k is the length of any side of the rectangle. Reasoning in the same way as we have done in Example 1.2 when deducing formula (1.24) for an electric dipole potential, we find that the distances $R_1 = |\mathbf{R}_1|$ extending from the middle of the first side and $R_3 = |\mathbf{R}_3|$ from the middle of the third side to the observation point a are

$$R_{1,3} = |\mathbf{R} \pm \mathbf{l}_2/2| \approx R \pm \mathbf{e}_R \mathbf{l}_2/2 \tag{2.69}$$

and that the distances from the middle of the second and of the fourth

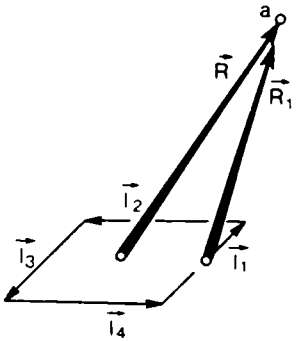


Fig. 2.7

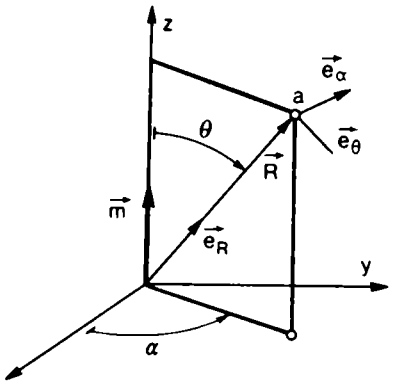


Fig. 2.8

side to the same point a are

$$R_{2,4} = |\mathbf{R} \mp \mathbf{l}_1/2| \approx R \mp \mathbf{e}_R \mathbf{l}_1/2 \tag{2.70}$$

According to (2.62b), the vector potential of the loop is given by

$$\begin{aligned} \mathbf{A} = (\mu_0 i/4\pi)\, [&\mathbf{l}_1/(R + \mathbf{e}_R \mathbf{l}_2/2) \\ &- \mathbf{l}_1/(R - \mathbf{e}_R \mathbf{l}_2/2) + \mathbf{l}_2/(R - \mathbf{e}_R \mathbf{l}_1/2) \\ &- \mathbf{l}_2/(R + \mathbf{e}_R \mathbf{l}_1/2)] \end{aligned} \tag{2.71}$$

After reducing to a common denominator, performing simple transformations of the numerator, and omitting small values of the order of l as against R, we get

$$\mathbf{A} = (\mu_0 i/4\pi)[[\mathbf{l}_2(\mathbf{e}_R \mathbf{l}_1) - \mathbf{l}_1(\mathbf{e}_R \mathbf{l}_2)]/R^2] \tag{2.72}$$

Using the known formula of triple vector product, the expression may be written as

$$\mathbf{A} = (\mu_0/4\pi R^2)\mathbf{m} \times \mathbf{e}_R \tag{2.73}$$

where $\mathbf{m} = i\mathbf{S}$, and $\mathbf{S}$ is the loop area equal to $\mathbf{l}_1 \times \mathbf{l}_2$.

Choosing the spherical coordinate system, where the angles θ are counted off from the direction of $\mathbf{m}$ (Fig. 2.8), we can come to a yet simpler expression:

$$A = A_\alpha = (\mu_0/4\pi R^2) m \sin\theta \tag{2.74}$$

The expression stresses the fact that A has only the αth component, which is of course apparent from (2.73) since the vector product $\mathbf{m} \times \mathbf{e}_R$ is in the direction of $\mathbf{e}_\alpha$.

Having taken the curl of the vector potential, determine now the components of the magnetic vector

$$\begin{aligned} \operatorname{curl} \mathbf{A} &= (\mu_0/4\pi) m \begin{vmatrix} \mathbf{e}_R/R^2 \sin\theta & \mathbf{e}_\theta/R \sin\theta & \mathbf{e}_\alpha/R \\ \partial/\partial R & \partial/\partial\theta & 0 \\ 0 & 0 & \sin^2\theta/R \end{vmatrix} \\ &= (\mu_0 \sqrt{m}/4\pi R^3)(2\cos\theta \cdot \mathbf{e}_R + \sin\theta \cdot \mathbf{e}_\theta) \end{aligned} \tag{2.75}$$

Comparing formulas (1.25) for the components of the loop field and of the elec-

tric dipole field

$$E_R = p \cos \theta / 2\pi\varepsilon_0 R^3, \quad E_\theta = p \sin \theta / 4\pi\varepsilon_0 R^3$$

and

$$B_R = \mu_0 m \cos\theta / 2\pi R^3, \quad B_\theta = \mu_0 m \sin \theta / 4\pi R^3$$

we can see why the vector $\mathbf{m} = i\mathbf{S}$ is called the moment of an electrokinetic or magnetic dipole.

Vector potential and magnetic flux. By Stokes' theorem

$$\int \text{curl } \mathbf{A} d\mathbf{S} = \oint \mathbf{A} d\mathbf{l} \tag{2.76}$$

But curl $\mathbf{A} = \mathbf{B}$, and hence the magnetic flux Φ can be expressed as a line integral of the vector potential

$$\Phi = \int \mathbf{B} d\mathbf{S} = \oint \mathbf{A} d\mathbf{l} \tag{2.77}$$

This expression can sometimes be useful for determining the vector potential.

Example 2.3. Inside a long magnetized core of radius r_0 there exists a magnetic induction parallel to the core axis, $B = B_z$. Outside the core (at $r > r_0$), the magnetic field is not present. Such a core is shown in Figs. 1.8 and 2.2.

Problem: (1) find the magnetic vector potential inside and outside the core by use of (2.77) and (2) verify by direct differentiation that curl $\mathbf{A}$ is equal to the specified value of $\mathbf{B}$.

Solution 1. Relying on the concept of symmetry, it is easy to conclude that $A = A_\alpha(r)$. So, taking the integral around the circumference (r = constant, z = constant), we find from (2.77) that at $r \leqslant r_0$

$$\Phi = B\pi r^2 = A \cdot 2\pi r \tag{a}$$

whence

$$A = A_\alpha = Br/2 \tag{b}$$

Similarly, at $r \geqslant r_0$

$$\Phi = B\pi r_0^2 = A \cdot 2\pi r \tag{c}$$

whence

$$A = A_\alpha = Br_0^2/2r \tag{d}$$

2. Calculating the curl in the cylindrical coordinate system gives

$$\text{curl } \mathbf{A} = \begin{cases} B\mathbf{e}_z \text{ at } r < r_0 \\ 0 \text{ at } r > r_0 \end{cases} \tag{e}$$

Conclusion. *Note* 1. In the general case the field vector may contain the components of which one is expressed in terms of the scalar potential and the other in terms of the vector potential:

$$\mathbf{F} = -\text{grad } \Psi + \text{curl } \mathbf{A} \tag{2.78}$$

To a particular class belong the fields in which one region has zero circulation and the other nonzero circulation. The field of Fig. 2.2 where curl $\mathbf{E} = 0$ outside the

magnetized core and curl $\mathbf{E} = -\partial\mathbf{B}/\partial t \neq 0$ inside the core may serve as an example. Here the electric field strength of the first region may be expressed as a gradient, though the value of scalar potential will prove ambiguous in the absence of a conditional (visualized) partition which does not allow drawing a closed loop to bound the second region.

In the absence of such a partition, the integral around the closed loop bounding the core of Fig. 2.2 becomes other than zero, though all contour elements lie in the region where curl $\mathbf{E} = 0$. Indeed, by Stokes' theorem in integral form (formula 3 in Appendix A3) applied in conjunction with Maxwell's second equation

$$\oint \mathbf{E}d\mathbf{l} = \int \operatorname{curl}\mathbf{E}d\mathbf{S} = -\int (\partial\mathbf{B}/\partial t)d\mathbf{S} = -\partial\Phi/\partial t \tag{2.79}$$

Applying a partition in the form of a half-plane (α = constant) to exclude bounding of the second region, we find that the integral taken along any closed path lying in the region with a zero curl is defined in a unique fashion.

Note 2. As is clear from (2.77), in fields with finite values of flux density $\mathbf{B}$ the tangential component of vector $\mathbf{A}$ on either side of the interface between any two regions is the same

$$A_{1\tau} = A_{2\tau} \tag{2.80}$$

i.e. does not undergo discontinuity. Otherwise, in tracing around the contour whose parallel sides are arbitrary close to each other but lie in different regions, we would obtain a finite value of the magnetic flux passing through a vanishingly small area, which contradicts the assumption of finiteness of $\mathbf{B}$.

Note 3. Both scalar and vector potentials determining the vector $\mathbf{E} = -\nabla\varphi - \partial\mathbf{A}/\partial t$ are measured in the units of the vector times the unit length. Thus the unit of a magnetic vector potential is that of $\mathbf{B}(\mathrm{T} = \mathrm{Wb/m^2})$ multiplied by a meter, i.e. Wb/m.

2.6. Electric Field Expressed in Terms of Two Potentials

The expression for an E field defined in terms of two potentials, which is also suitable for E fields with nonzero curls, represents a generalization. The conclusion of exceptional importance is drawn by performing very simple mathematical operations, though using powerful means of vector analysis.

Let us focus on Maxwell's second equation

$$\operatorname{curl}\mathbf{E} = -\partial\mathbf{B}/\partial t$$

and express here the magnetic flux density as the curl of a vector potential. Keeping in mind that the succession of differentiation with respect to time and spatial coordinates is commutative, we obtain

$$\operatorname{curl}\mathbf{E} = -(\partial/\partial t)\operatorname{curl}\mathbf{A} = -\operatorname{curl}(\partial\mathbf{A}/\partial t)$$

or

$$\operatorname{curl}(\mathbf{E} + \partial\mathbf{A}/\partial t) = 0 \tag{2.81}$$

(the sum of curls is equal to the curl of the sum because the differential operation is linear). The solution to the latter equation may be given as

$$\mathbf{E} + \partial\mathbf{A}/\partial t = -\operatorname{grad}\varphi \tag{2.82}$$

since the curl of any gradient is identically zero.

This solution is in agreement with Maxwell's equation for an electric field:

$$\mathbf{E} = -\nabla\varphi - \partial\mathbf{A}/\partial t \tag{2.83}$$

which says that *the electric field strength is equal to the sum of an electric potential gradient and the time derivative of a magnetic vector potential.*

This remarkable equality reflects the differential expression of the law of electromagnetic induction which, in Faraday's opinion, was determined by the time variation of an electrotonic condition.

Indeed, integrating (2.83) around the closed path yields

$$\oint \mathbf{E}d\mathbf{l} = \mathscr{E} = -\oint \nabla\varphi d\mathbf{l} - (\partial/\partial t)\oint \mathbf{A}d\mathbf{l} \tag{2.84}$$

But the first integral on the right side is identically equal to zero and the second, according to (2.77), is equal to a magnetic flux. The obtained expression is thus identical to the Faraday-Maxwell law:

$$\mathscr{E} = -\partial\Phi/\partial t$$

Formula (2.83) enables us to determine the induced electric field strength at any observation point, i.e. the component resulting from the varying magnetic field, whereas formula (2.84) can only be useful for determining an integral quantity, namely, the electromotive force.

It is essential to pay attention to the fact that the induced field $\mathbf{E}_A = -\partial\mathbf{A}/\partial t$ does not depend directly on the magnetic field at the same point. Thus, there are regions where $\mathbf{B} = 0$, whereas the induced electric field is nonzero. On the other hand, one can detect the points where the induced electric field is zero despite the presence of a time-varying magnetic field. An example is a two-wire overhead line.

For a two-wire line, the magnetic flux density can be determined from the circuital law by calculating the field vectors for each wire separately and then adding them together by the principle of superposition. In the plane of wires (Fig. 2.9), the magnetic flux density between the wires, where $x^2 < a^2$, is given by

$$B = B_y = (\mu_0 i/2\pi)\,[1/(a - x) + 1/(a + x)] \tag{2.85}$$

In the same plane, the vector potential according to (2.77) is

$$A = A_z = (\mu_0 i/2\pi)\ln[(a + x)/(a - x)] \tag{2.86}$$

The same result can be obtained in a more general way (the superposition principle) by summing up the values of the expressions for a vector potential of the left and the right wire. Each of these

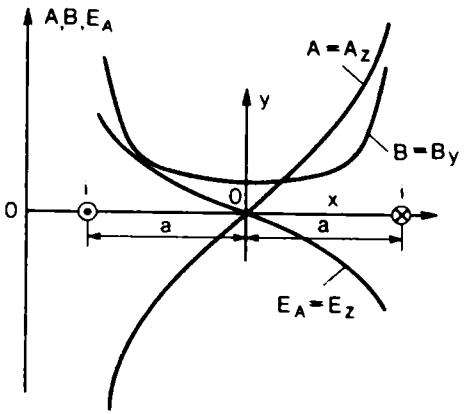


Fig. 2.9

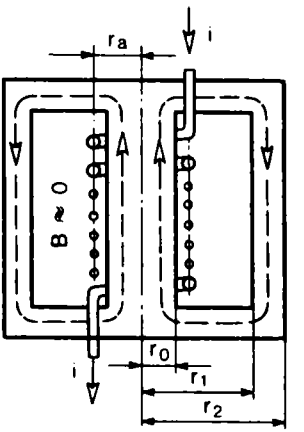


Fig. 2.10

expressions agrees with formula (2.68) derived for an insulating gap in the coaxial cable, where the magnetic field is in the direction of the field of one wire; one must take into account here the difference in r between the left and the right wire and also the opposite direction of current flows.

The graphs of corresponding quantities plotted for a certain moment of time t appear in Fig. 2.9, which also shows the graph of the induced field $E_A = -\partial A/\partial t$ (in the plane $y = 0$) that has only the zth component. The field E_A is equal to zero at $x = y = 0$, but $B \neq 0$ and $\partial B/\partial t \neq 0$. Along with the E_A field, the potential component $E_\varphi = -\nabla\varphi$ due to wire charges exists between the wires. In the plane of wires, E_φ generally has a dominant xth component. The direction of this component is from right to left if the line transfers energy out of the plane of the paper, or from left to right if the flow of energy is into the paper.

As follows from the analysis made in deducing (2.86), the vector potential of the two-wire line of Fig. 2.9 at any point a distance $r_\odot$ remote from the left wire (the current moves along the z axis) and a distance $r_\otimes$ from the right wire (the same current but opposite to the direction of the z axis) is given by

$$A = A_z = (\mu_0 i/2\pi) \ln (r_\otimes/r_\odot) \tag{2.87}$$

In Fig. 2.9, the values of A, B, and E_A are the functions of x at $y = 0$.

A transformer represents another interesting example. For simplicity, visualize a transformer as the one having an internal cylindrical core ($r = r_0$) whose magnetic flux flows through the yoke and closes on itself as it passes through an outer coaxial cylinder with an internal radius r_1 and external radius r_2. Fig. 2.10 illustrates such a transformer (sometimes called shell-type) with a primary whose turns have a radius r_a. In the region surrounded by the magnetic circuit, $r_a < r < r_1$, the magnetic field strength is very small if the ferromagnetic material shows high permeability, and

so $B \approx 0$. If the magnetic flux density in the internal core ($r_0 = 2$ cm) is $B = 1$ T, then at $r = 2.5$ cm we have according to (2.77):

$$A = A_\alpha = B\pi r_0^2/2\pi r = 8 \times 10^{-3} \text{ T m} = 8 \times 10^3 \text{ G cm}$$

If the field decays by one half in 10^{-3} s, the electric field strength is

$$E = E_\alpha = -\partial A/\partial t = 4 \text{ T m/s} = 4 \text{ V/m}$$

because 1 T = 1 V s/m^2 and 1 T m = 1 V s/m. This quantity is determined by the rate of change of the vector potential and is practically independent of a weak magnetic field in this region.

If the internal core winding of radius $r = 2.5$ cm contains, say, 1 000 turns, its field strength E_α will be bound as an integral taken over the entire turns, and the emf per turn will be 0.63 V. Since the integral $\int \mathbf{E}d\mathbf{l}$ rises by 0.63 V with each full turn, the emf for 1000 turns will become equal to 630 V.

Components due to scalar and vector potentials. An unambiguous resolution of the **E** field into the components $\mathbf{E}_\varphi = -\nabla\varphi$ and $\mathbf{E}_A = -\partial\mathbf{A}/\partial t$ presents sometimes difficulties. This is particularly the case where the analysis involves not the entire field but only a portion of it enclosed in space within a certain region. Notice that these difficulties are generally not met in calculating the vector **E** or its divergence and curl, the physical meaning of which is uniquely expressed by Gauss' theorem and Maxwell's second equation. As a matter of fact, such integral quantities as a magnetic flux Φ and voltage U remain unambiguous.

In the simple examples that follow we can definitely specify the condition at which the potential component vanishes or arises and also calculate the values of $-\partial\mathbf{A}/\partial t$ and $-\nabla\varphi$.

Example 2.4. A long magnetized core of radius r_0 (see Figs. 1.8 and 2.2) is coaxially surrounded by a wire ring (Fig. 2.11) of radius $r = 2\text{ cm} > r_0 = 1$ cm and of cross-sectional area $S = 1$ mm^2. One half of the ring is made of copper wire with $\sigma_1 = 56 \times 10^4$ S/cm, and the other of lead wire with $\sigma_2 = 4.55 \times 10^4$ S/cm.

The magnetic flux density in the core uniformly vanishes: $\partial B/\partial t = -100$ T/s = constant.

Of course, these conditions exist for a short time, 10 to 20 ms, but still we can take the current induced in the ring to be constant during this period or at least a fraction of a millisecond after B begins to change.

Problem 1. Determine the electric field strength in copper and lead and separate the components corresponding to the vector and the scalar potential.

Problem 2. Explain the physical cause that gives rise to the component $-\nabla\varphi$.

Solution 1. It is obvious that the emf

$$\mathcal{E} = -\pi r_0^2 \partial B/\partial t = E_\alpha \cdot 2\pi r = 31.4 \text{ mV}$$

The loop resistance

$$R = (\pi r/S)(1/\sigma_1 + 1/\sigma_2) = 14.95 \times 10^{-3}\Omega$$

The current set up in the ring is $i = \mathcal{E}/R = 2.1$ A.

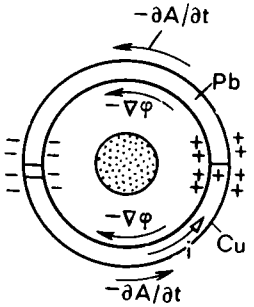


Fig. 2.11

At this current, the field strength in copper

$$E_1 = i/S\sigma_1 = 0.375 \text{ mV/cm}$$

and in lead

$$E_2 = i/S\sigma_2 = 4.61 \text{ mV/cm}$$

But the sinusoidally varying component of the vector potential is only the function of changes of the magnetic flux density in the core. Applying (2.77) to derive the expression for vector potential $A = A_\alpha = B\pi r_0^2/2\pi r$, we find that

$$E_A = -\partial A/\partial t = 2.50 \text{ mV/cm}$$

both in the copper and in the lead half of the ring. The resultant difference between E_1 and E_2 can be attributed only to the appearance of the gradient of a scalar potential:

$$\begin{aligned} E_1 &= -\partial A/\partial t - (\nabla\varphi)_{\alpha 1} \\ E_2 &= -\partial A/\partial t - (\nabla\varphi)_{\alpha 2} \end{aligned} \qquad \text{(a)}$$

The obligatory condition

$$\oint \nabla\varphi d\mathbf{l} = (\nabla\varphi)_{\alpha 1} l_1 + (\nabla\varphi)_{\alpha 2} l_2 = 0 \qquad \text{(b)}$$

leads us to conclude that

$$(\nabla\varphi)_{\alpha 2} = -(\nabla\varphi)_{\alpha 1} l_1/l_2 \qquad \text{(c)}$$

At $l_1 = l_2$, we have

$$(\nabla\varphi)_{\alpha 2} = -(\nabla\varphi)_{\alpha 1} = (\nabla\varphi)_\alpha \qquad \text{(d)}$$

Given this condition, from Eqs. (a) we obtain the component due to the scalar potential gradient

$$-(\nabla\varphi)_\alpha = -(\nabla\varphi)_{\alpha 2} = (E_2 - E_1)/2 = 2.12 \text{ mV/cm}$$

2. The physical explanation of the result is very simple. In the junction region where the current flows from copper into lead (at $\sigma_1 > \sigma_2$) the distribution of positive charges takes place, and in the junction region where the current flows from lead into copper, the distribution of negative charges occurs, as schematically shown in Fig. 2.11. These charges, acting by Coulomb's law, produce the field in lead that is in the direction of the component $-\partial A/\partial t$, and so the field grows. In copper, the same charges produce the field in the opposite direction to the component $-\partial A/\partial t$ and hence the field in copper weakens.

What will the voltmeter read? This question often arises if the voltmeter connecting wires lie in the region of a changing vector potential ($-\partial A/\partial t \neq 0$), i.e. if an induced electric field exists in the wires.

Imagine for more clarity that the voltmeter at hand is an electrostatic device and shows the potential difference between its electrodes (as the thermometer shows its

own temperature), regardless of whether or not this potential difference conforms to that between the points to which the voltmeter is connected. Proceeding from the above, it is easy to provide the answer to the question raised after working through the following example.

Example 2.5. Two voltmeters V_1 and V_2 (Fig. 2.12) measure potential difference under the conditions given in Example 2.4.

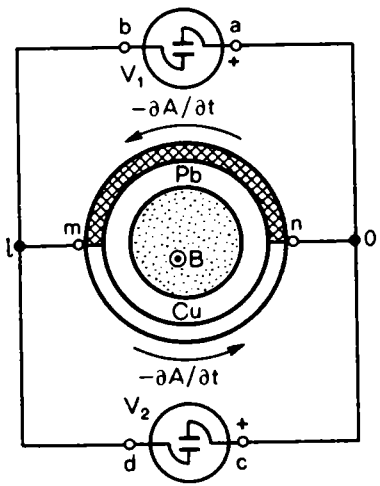


Fig. 2.12

The points of connection of the voltmeters to the ring, *m* and *n*, are diametrically opposite and coincide with the points of junction between the lead and copper wires. Assume that the vector **B** goes into the plane of the page and $\partial B/\partial t < 0$; this is consistent with the direction of $-\partial A/\partial t = E_A$ shown in the figure (this component is in the positive direction of $A = A_\alpha$ at $B = B_z > 0$ since $\partial A_\alpha/\partial t < 0$)[1].

Problem 1. What value will each voltmeter show? 2. How can we explain the difference, if any, in the readings of the two voltmeters?

The electrostatic voltmeter of a slightly elaborate design can also indicate the sign of potential difference at its terminals. Keeping this in mind, we have to give the answer with due regard for the sign and thus assume that

$$U_1 = \varphi_a - \varphi_b,\ U_2 = \varphi_c - \varphi_d$$

These conditions correspond to the "plus" sign marked near one of the terminals of each voltmeter; this sign denotes that the deflection on the voltmeter is positive when the potential at the "plus" terminal is higher than that at the other terminal.

Solution. 1. Evaluating the integral around the closed path, *a, b, l, m,* Pb, *n, O, a* (see Fig. 2.12), we obtain

$$\oint \mathbf{E} d\mathbf{l} = \varphi_a - \varphi_b - E_2 \cdot \pi r = 0$$

where E_2 is the field strength in lead.

---

[1] Check that the **E** direction is in agreement with Lenz' law: a change in magnetic flux causes an emf that tends to induce a current opposing the change in flux.

This result is obvious because the connecting wires do not carry current and hence $E = 0$ in the regions *bl*, *lm*, *nO*, and *Oa*. The reason why the complete integral is equal to zero is that the varying magnetic flux does not link the loop.

It is known from the solution of the problem stated in Example 2.4 that $E_2 = 4.61$ mV/cm at $r = 2$ cm. Therefore the solution sought is

$$U_1 = \varphi_a - \varphi_b = E_2 \cdot \pi r = 29.0 \text{ mV}$$

To determine the readings of the second voltmeter, we integrate around the loop, *c, d, l, m,* Cu, *n, O, c:*

$$\oint E d\mathbf{l} = \varphi_c - \varphi_d + E_1 \cdot \pi r = 0$$

whence, knowing the solution to the problem formulated in Example 2.4, we get

$$U_2 = \varphi_c - \varphi_d = -E_1 \cdot \pi r = -2.4 \text{ mV}$$

where E_1 is the field strength in copper.

2. The difference in the readings of two voltmeters is easy to find by extending the integral of **E** along the external loop, *a, b, l, d, c, O, a:*

$$\oint \mathbf{E} d\mathbf{l} = (\varphi_a - \varphi_b) - (\varphi_c - \varphi_d) = U_1 - U_2$$

This integral, however, is nonzero because the loop under examination is linked by the varying flux; in tracing around the loop counterclockwise, the flux flowing into the page should be considered positive. Hence,

$$\oint \mathbf{E} d\mathbf{l} = U_1 - U_2 = -(\partial / \partial t)(B\pi r_0^2) = \mathscr{E}$$

From the above calculations (see Example 2.4), $\mathscr{E} = 31.4$ mV, which fully agrees with the found difference between the readings of the voltmeters.

It is quite possible to formulate these problems in a variety of ways and work them by the method presented here. Where moving bodies are involved, substantial difficulties arise in calculations.

Chapter 3 ELECTROMAGNETIC FIELD IN POLARIZABLE BODIES AND CONDUCTORS. STATIC AND QUASISTATIC CONDITIONS

3.1. Stationary (Quasistatic) Field in the Presence of Polarized Bodies

Electric polarization vector. The electric field of an uncharged body may be different from zero if the body contains pairwise bound charges $\pm q$ that produce, on the average, a non-zero electric moment

$$\sum q\mathbf{l} = \sum \mathbf{p} \neq 0 \text{ at } \sum q = 0$$

Conducting bodies immersed in an external electric field also exhibit an electric moment. Fig. 3.1 shows the resultant field after a conducting body has been placed into an external uniform electric field. The electric field exerts force on charges capable of moving inside the conductor. They move in the field direction until the field inside the conductor cancels. *In static conditions, the electric field inside a conductor is equal to zero* as are the volume charge and dipole moments.

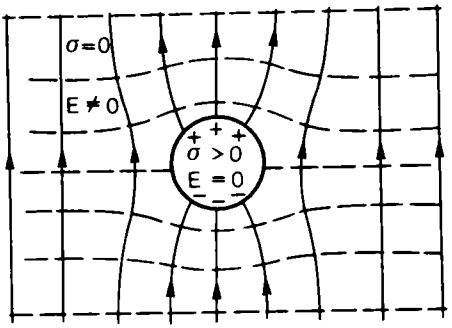


Fig. 3.1

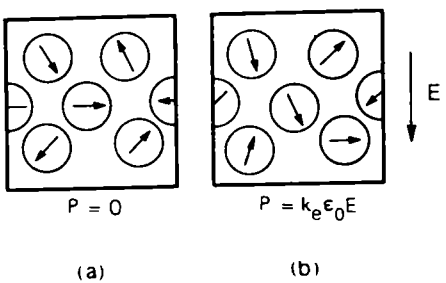


Fig. 3.2

It is different with dielectrics. Under static or almost static conditions, these materials can have an electric field, space charge, and also bound charges exhibiting a dipole moment. Neutral molecules of a substance contain pairwise bound charges. The sum of charges is equal to zero, but the mean positions of charges may not be coincident, with the result that neutral molecules can possess an electric moment. Such molecules become dipoles. If a substance shows an ordered orientation of dipoles, the substance is said to be polarized.

A measure of intensity of electric polarization is the total dipole moment per unit volume (the density of dipole moments):

$$\mathbf{P} = \lim_{v \to 0} (\sum \mathbf{p}/V) = \langle \mathbf{p} \rangle = \langle q\mathbf{l} \rangle \tag{3.1}$$

Here, the summation sign covers all the dipoles in volume V, and the symbol $\langle\ \rangle$ denotes averaging related to a unit volume.

The vector $\mathbf{P}$ is the intensity of polarization, or just polarization.

Figure 3.2*a* illustrates randomly oriented dipole molecules, and so $P = 0$. In Fig. 3.2*b* are shown the same molecules exhibiting, in the average, an ordered component of moment $\mathbf{P}$ in the $\mathbf{E}$ field direction.

Dipoles have an electric field. A polarized substance exhibits an electric moment. For a volume element dv, this moment may be expressed in terms of polarization:

$$d\mathbf{p} = \mathbf{P}dV \tag{3.2}$$

Since the electic field of dipoles is nonzero and defined by (1.24) and (1.25) in terms of the dipole moment, the field of a polarized substance must also be nonzero; the potential of this field can be determined from (1.24) by summing up the effect of all elementary dipole moments, $\mathbf{P}dV$. The sought-for potential is thus the integral

$$\varphi_P = \int (\mathbf{P}\mathbf{e}_R / 4\pi\varepsilon_0 R^2) dV \tag{3.3}$$

extended over the entire volume where $\mathbf{P} \neq 0$.

The density of bound charges in a polarized medium. A simple analysis of the integrand in (3.3) gives significant results. From the vector analysis (see Appendix A5) it follows that

$$\mathbf{e}_R / R^2 = \nabla(1/R) \tag{3.4}$$

if the vector **R** is drawn from the variable source point (x, y, z) to the fixed observation point (x', y', z'),

$$R = \sqrt{(x' - x)^2 + (y' - y)^2 + (z' - z)^2}$$

and the differentiation (expressed in terms of the operator ∇) is performed with respect to the coordinates of the source point (x, y, z). If we were to differentiate with respect to the observation point coordinates (x', y', z'), the result would be of the opposite sign.

Taking into account (3.4), we can express the integrand in (3.3) in the form

$$\mathbf{P}\mathbf{e}_R/R^2 = \mathbf{P}\nabla(1/R) \tag{3.5}$$

But by the rule of differentiation of products, as applied to **P** and $1/R$,

$$\nabla[\mathbf{P}(1/R)] = (1/R)\nabla\mathbf{P} + \mathbf{P}\nabla(1/R) \tag{3.6}$$

and, hence,

$$\mathbf{P}\mathbf{e}_R/R^2 = \mathbf{P}\nabla(1/R) = \nabla(\mathbf{P}/R) - (\nabla\mathbf{P})/R \tag{3.7}$$

Substituting this result in (3.3), we see that

$$\varphi_P = \int (\mathbf{P}\mathbf{e}_R/4\pi\varepsilon_0 R^2)dV = \int (-\operatorname{div}\mathbf{P}/4\pi\varepsilon_0 R)dV \tag{3.8}$$

The last integral should be extended over the entire volume occupied by the polarized medium. In the final expression, we discard the integral of the first term on the right of (3.7) as being equal to zero. Indeed, applying the Ostrogradsky-Gauss theorem to this integral gives

$$\int \operatorname{div}(\mathbf{P}/R)dV = \oint (\mathbf{P}/R)d\mathbf{S} \tag{3.9}$$

The integral must be extended over the entire volume of the polarized medium or even over a larger volume, which is permissible from the statement of the problem proper: the inclusion of extra volume elements into the integral of (3.8) adds nothing but zero. But extending the integration beyond the boundaries of the region, $\mathbf{P} \neq 0$, reduces the integrals of (3.9) to zero because the closed surface S lies in the region where $\mathbf{P} = 0$.

Comparing expression (3.8)

$$\varphi_P = \int (-\operatorname{div}\mathbf{P}/4\pi\varepsilon_0 R)dV$$

with the earlier deduced expression (1.21a)

$$\varphi = \int (\rho/4\pi\varepsilon_0 R)dV \tag{3.10}$$

allows us to assume that for polarized bodies, the electric field is due only to an unbalanced charge caused by nonuniform polarization; the density of this charge is then

$$\rho_P = -\operatorname{div}\mathbf{P} \tag{3.11}$$

This quantity is called the density of a bound charge, ρ_b. The validity of the above assumption is clearly evident from simple examples.

In a uniformly polarized medium, positive and negative charges of dipoles over the

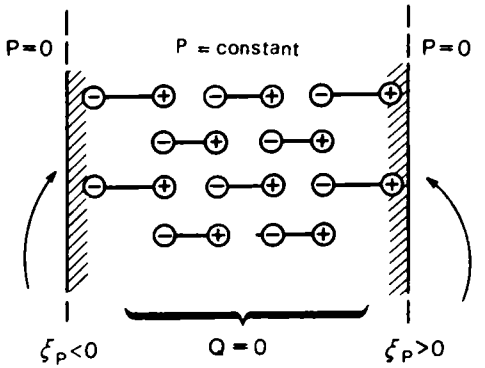


Fig. 3.3

entire bulk of the substance mutually cancel out (Fig. 3.3); the uncompensated charges are seen to lie only at the boundary of the polarized substance, so that the volume density of bound charges is equal to zero. In this case, only the surface density of bound charges is evident:

$$\xi_b = \xi_P = \mathbf{P}\mathbf{e}_n = P_n \tag{3.12}$$

But a nonuniformly polarized substance can have volume bound charges as well as surface bound charges. If, for example, polarization varies linearly

$$P = P_x = a + bx$$

a negative volume charge ρ_b appears in the space. Indeed, we may visualize the rise in polarization as the growth of the number of dipoles in a unit volume. In this case, the number of negative charges in the entire region where $P \neq 0$ is greater than the number of positive charges (Fig. 3.4).

The use of the Ostrogradsky-Gauss theorem and the operation div presupposes that the vector **P** being differentiated is continuous. To satisfy the condition of continuity at the boundaries too, we can assume that near the real boundary of a dielectric the vector **P** changes but gradually in a thin boundary layer.

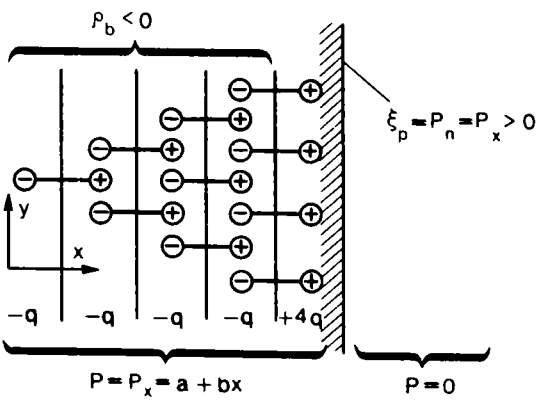


Fig. 3.4

If the vector becomes discontinuous at, say, the interface of two media, we can pass to the so-called surface divergence that defines the surface density of a charge due to polarization,

$$\xi_P = -\mathrm{Div}\,\mathbf{P} = -(\mathbf{P}_2 - \mathbf{P}_1)\mathbf{e}_n = P_{1n} - P_{2n} \tag{3.13}$$

where $\mathbf{P}_1$ and $\mathbf{P}_2$ are the intensities of polarization on both sides of the discontinuity surface.

The density of surface charges, like that of point charges (see Example 2.2), should also be allowed for when performing integration by use of (1.21b).

Electric flux density (electric displacement). As mentioned above, the electric field of a polarized medium may be regarded as a field of bound charges whose density

$$\rho_P = -\mathrm{div}\,\mathbf{P} \text{ and } \xi_P = -\mathrm{Div}\,\mathbf{P}$$

It is only the existence of these charges that distinguishes the fields in the presence of polarized media. Therefore, all the field equations valid for vacuum hold well if they take into account bound charges along with free charges.

The above statement leads us to introduce an additional term for space charges into the right side of Gauss' theorem:

$$\mathrm{div}\,\varepsilon_0\mathbf{E} = \rho + \rho_P \tag{3.14}$$

Expressing the bound charge density as the divergence of polarization, we find that

$$\mathrm{div}\,\varepsilon_0\mathbf{E} = \rho - \mathrm{div}\,\mathbf{P} \tag{3.15}$$

Assuming the divergence operation to be linear, we arrive at a new statement of Gauss' theorem

$$\mathrm{div}(\varepsilon_0\mathbf{E} + \mathbf{P}) = \rho \tag{3.16}$$

What distinguishes this statement is that on the right of the equality there is only the volume density of an electric charge in the common sense of the word (free charge), the effect of the polarized medium being taken account of by the summand $\mathbf{P}$ under the sign of divergence.

The sum of vectors, the divergence of which is equal to the density of a free charge, is known as the vector of electric flux density, or the vector of electric displacement density [1]:

$$\mathbf{D} = \varepsilon_0\mathbf{E} + \mathbf{P} \tag{3.17}$$

Introducing this vector, we obtain field equations

$$\begin{gathered}\mathrm{div}\,\mathbf{D} = \rho \\ \oint \mathbf{D}d\mathbf{S} = \int \rho dV = Q\end{gathered} \tag{3.18}$$

They contain in explicit form only the free charge Q and the volume density of the

[1] The term *displacement* arises from the former conception which comes to the following: an electric field causes two similar processes: displacement of electricity in vacuum ($\varepsilon_0\mathbf{E}$) and in a substance ($\mathbf{P}$).

free charge, ρ, if we can disregard point and surface charges [see (1.21b) and Example 2.2)].

Polarization and electric susceptibility. Field theory most often deals with electric polarization in dielectrics and poor conductors that occurs under the action of an electric field: forces $+q\mathbf{E}$ and $-q\mathbf{E}$ disturb the mean position of pairwise bound charges ($+q$ and $-q$) in molecules. The field displaces positive and negative charges in opposite directions and thus gives rise to polarization expressed as the mean electric moment per unit volume.

In molecules which have no moment in the absence of a field (nonpolar molecules), moments may appear in the direction parallel to the applied field **E** as a result of displacement of the mean position of orbital electrons. If molecules exhibit moments in the absence of a field (polar molecules), but have the mean moment equal to zero because of random orientation, the applied field tends to orient the moments of molecules in the field direction.

In both cases polarization depends on the field applied. But it is easy to see that the intensity of polarization of polar and nonpolar molecules must depend differently on such physical factors as temperature, electric field frequency, etc.

Really, thermal motion must favor random orientation of polar molecules; as the temperature rises (while the electric field remains the same) the substance polarization due to polar molecules must decrease. On the contrary, the polarization of nonpolar molecules must be independent of temperature.

Polar molecules subjected to an electric field take a certain time to align in the field direction. Therefore, the polarizability of a substance containing polar molecules may markedly decrease with an increased frequency of the applied field (see Sec. 3.2). Last, polar molecules may come to "saturation": in a certain sufficiently strong field, but of course not too high to bring the dielectric to rupture, almost all the dipoles of molecules may be found to be aligned along the field and a further rise in field strength does not lead to any noticeable increase in the degree of polarization.

The polarization of a substance having nonpolar molecules does not depend on frequency over a very wide frequency range and is proportional to the field strength until the dielectric breaks down. The field strength at which the dielectric ruptures is known as breakdown strength.

In the simplest case, polarization is proportional to a field strength (Fig. 3.5*a*):

$$\mathbf{P} = \varepsilon_0 k_e \mathbf{E} \tag{3.19}$$

where k_e is a proportionality factor called electric susceptibility.

In the centimeter-gram-second (CGS) system, the electric susceptibility is the factor χ entering the expression

$$4\pi\mathbf{P} = 4\pi\chi\mathbf{E} = k_e\mathbf{E} \tag{3.19a}$$

The factor χ is related to k_e, adopted in the SI system, by the equality $4\pi\chi = k_e$. All the tables of physical parameters generally give the factor χ.

Expression (3.19), like all the text of this section, relates to steady-state conditions. The *P-E* relation under transient conditions and in ac fields is discussed in Secs. 3.2, 5.5, and 5.6.

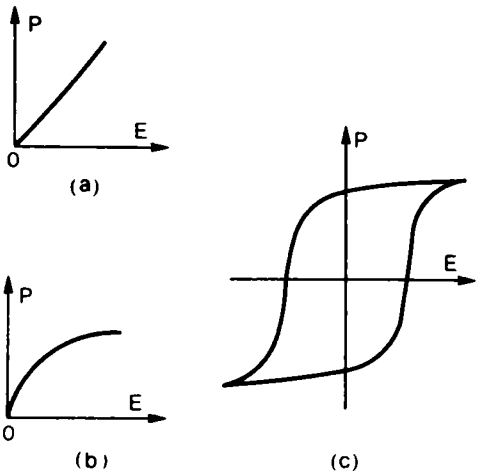


Fig. 3.5

Polarization is not always proportional to field strength; it may vary nonlinearly with E (Fig. 3.5*b*) and even in a hysteretic manner (Fig. 3.5*c*). The dielectric materials which exhibit a hysteresis loop are known as ferroelectrics, or ferroelectric materials. An example is the Rochelle salt (potassium sodium tartrate) whose *P-E* relation resembles the magnetic characteristic of ferromagnets, or ferromagnetic materials. It is important to note that polarization in ferroelectrics may exist even in the absence of an E field ($P \neq 0$ at $E = 0$).

For ordinary solid and liquid dielectrics, the susceptibility lies in the order of a few units: for transformer oil, $k_e = 1$, and for porcelain, $k_e = 4$. In ferroelectrics, k_e depends on field strength and may range into the hundreds and even thousands. For air and most of the gases, k_e is by far lower than unity and may be taken to equal zero in practical calculations.

In anisotropic substances (crystals and dielectrics with an ordered structure), the susceptibility is a function of the field direction. In an anisotropic medium, therefore, the vectors **P** and **E** may not be in the same direction; for such a medium the susceptibility is expressed in terms of a tensor.

Of great importance is the fact that an electric field is not the only source of polarization of a dielectric. Electrical engineering widely uses the effect of polarization occurring in some crystals when subjected to a mechanical stress. A well known example is a crystal (piezoelectric) phonograph pickup: as the stylus moves along the plate, it causes vibrations and thus a change in the compression of a small crystal piece. The compression gives rise to an electric polarization; in the final analysis, a varying electric field due to polarization exerts the desired effect on an electron tube or transistor. Piezoelectric crystals (quartz) are also suitable for measuring the pressure of impacts, blasts, etc.

In turn, the polarization caused by an electric field results in mechanical deformation of a crystal (the electrostrictive effect). Electrostriction underlies the operation of quartz resonators intended to control frequency, for example.

Dielectric permittivity. Proceeding from the general statement of electric displace-

ment and the assumption that **P** is proportional to **E**, we can readily pass to the equality

$$\mathbf{D} = \varepsilon_0 \mathbf{E} + \mathbf{P} = \varepsilon_0 (1 + k_e)\mathbf{E} = \varepsilon_0 \varepsilon \mathbf{E} \tag{3.20}$$

where $\varepsilon = 1 + k_e$ is the dielectric constant, or relative permittivity.

In the above equality, the first part, $\mathbf{D} = \varepsilon_0 \mathbf{E} + \mathbf{P}$, is always valid, whereas the second part is true only if **P** is in proportion to **E**. Thus, there is no sense in defining ε when polarization arises from a mechanical stress.

In the SI system, in use is the product $\varepsilon_0 \varepsilon$ called the *absolute capacitivity*, or *absolute dielectric constant*:

$$\varepsilon_a = \varepsilon_0 \varepsilon$$

Substituting **D** given by (3.20) into Gauss' theorem (3.18) yields

$$\oint \mathbf{D}\, d\mathbf{S} = \oint \varepsilon \varepsilon_0\, \mathbf{E} d\mathbf{S} = \int \rho dV = Q \tag{3.21}$$

If a point charge q lies in an unbounded medium of permittivity ε, its field strength is

$$E = E_R = q/4\pi\varepsilon\varepsilon_0 R^2 \tag{3.22}$$

as is readily apparent from the considerations of symmetry. Thus, the factor ε appears in the denominator of Coulomb's law.

Field vectors at the interface between two media. Boundary conditions [1]. The surface density of a free charge, ξ, on a conductor adjacent to a dielectric and the normal component of a field strength, E_n, on its surface (Fig. 3.6) in the static case are related to each other by the equation

$$\xi = D_n \tag{3.23}$$

where the normal points to a dielectric and $\mathbf{D} = \varepsilon \varepsilon_0 \mathbf{E}$ for a linear dielectric.

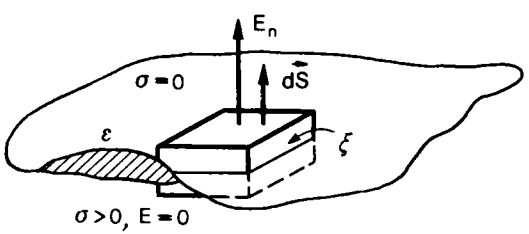


Fig. 3.6

The above statement directly follows from Gauss' theorem (3.21) applied to a closed surface covering the boundary between a conductor (where $E = 0$) and a dielectric (see Fig. 3.6).

Indeed, let the surface be a rectangular parallelepiped with bases dS parallel to the conductor-dielectric interface and the parallelepiped height be negligible. In this case

[1] All the text here is set forth on the assumption that no moving bodies are present. This fact should be given due consideration in discussing boundary conditions whose formulation is sometimes different for moving boundaries even at low (prerelativistic) speeds.

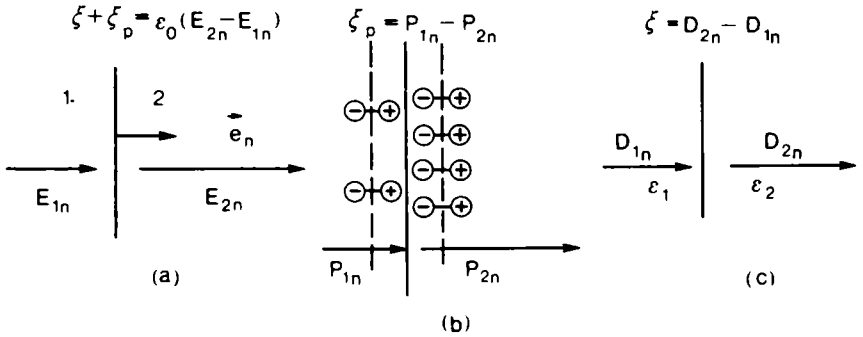


Fig. 3.7

the outward flux through the surface of the parallelepiped is equal to the flux leaving its upper face:

$$d\Psi = \mathbf{D}d\mathbf{S} = \varepsilon\varepsilon_0 E_n dS$$

because the field does not exist inside the conductor in statics. The charge enclosed within the surface is equal to the surface charge density ξ on the conductor times the conductor area dS within the parallelepiped:

$$dQ = \xi\, dS$$

Formula (3.23) directly follows from the comparison of the last two equations.

Consider now the interface of two dielectrics. The general principle underlying Lorentz' field theory for dielectrics states that, taking into consideration both free and bound charges, we can express all field laws in terms of field strength vectors, the permittivity of a medium being ignored and displacement vectors disregarded. The total surface charge at the interface between two media (Fig. 3.7*a*) is then given by

$$\xi + \xi_P = \varepsilon_0(E_{2n} - E_{1n}) \tag{3.24}$$

But according to (3.13) the surface charge density due to polarization (Fig. 3.7*b*) is

$$\xi_P = P_{1n} - P_{2n} \tag{3.25}$$

Comparing the last two equalities gives the free surface charge (Fig. 3.7*c*)

$$\xi = \varepsilon_0 E_{2n} + P_{2n} - (\varepsilon_0 E_{1n} + P_{1n}) = D_{2n} - D_{1n} \tag{3.26}$$

In the absence of a free surface charge, $\xi = 0$, the normal components of $\mathbf{D}$ do not undergo discontinuity, and the boundary condition takes the form

$$\varepsilon_2 E_{2n} = \varepsilon_1 E_{1n} \quad \text{or} \quad D_{2n} = D_{1n} \tag{3.27}$$

The normal component of E can be discontinuous across the interface. On the contrary, the tangential component of an electric field is continuous across the boundary between two media:

$$E_{2t} = E_{1t} \tag{3.28}$$

The latter condition is obtained by applying Maxwell's second equation to a small rectangular loop (Fig. 3.8) whose both sides Δl are parallel to the interface and lie in

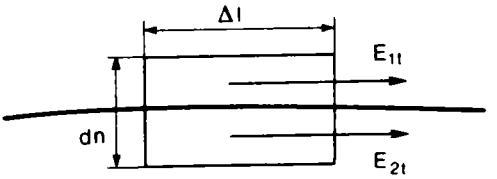


Fig. 3.8

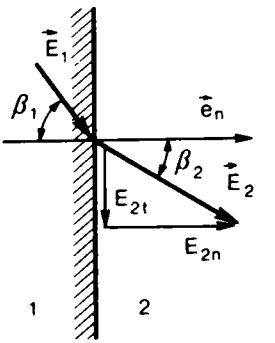


Fig. 3.9

different media, the other two sides dn of the loop being negligible as compared to a value of Δl. The integral of the field strength along the loop, assuming the field stationary, is then

$$\oint \mathbf{E} d\mathbf{l} = (E_{2t} - E_{1t})\,\Delta l = 0 \tag{3.29}$$

Hence, $E_{2t} - E_{1t} = 0$.

Note also that the tangential component is a vector "normal to the normal", and can have two components in the plane tangent to the interface. In this case it is safe to assume that (3.28) is applicable to any of the two components. Expression (3.28) then also holds for vectors $\mathbf{E}_{1t} = \mathbf{E}_{2t}$.

In the presence of a varying magnetic field, the right part of the integral in (3.29) is equal to $-\partial\Phi/\partial t = -(\partial B_\eta/\partial t)\Delta l dn$. Hence,

$$E_{2t} - E_{1t} = -(\partial B_\eta/\partial t) dn \to 0$$

at any finite value of $\partial B_\eta/\partial t$. Here B_η is the magnetic flux density component normal to the plane of a loop with sides Δl and dn.

Equalities (3.27) and (3.28) lead us to conclude that in isotropic media the field vectors $\mathbf{E}_1$ and $\mathbf{E}_2$ and the normal $\mathbf{e}_n$ going from medium *1* into medium *2* form angles β_1 and β_2 (Fig. 3.9), for which

$$\tan\beta_1/\tan\beta_2 = (E_{1t}/E_{1n})/(E_{2t}/E_{2n}) = E_{2n}/E_{1n} = \varepsilon_1/\varepsilon_2 \tag{3.30}$$

if there is no free surface charge at the interface.

The effect of conductivity on the field in a dielectric. Practical dielectrics always display conduction to a certain degree. If the conductivity of a medium is not too high to screen the effects observed in polarization, the medium is said to be an imperfect dielectric, or poor conductor; otherwise, the medium is said to be a conductor. Conduction heavily affects the field distribution. But in a dc field under steady-state conditions, the conductivity of a medium does not play any role — the field is definable as a field in a conductor (conducting medium). For many dielectrics, however, such steady-state conditions set in a few hours after applying a dc field.

3.2. A Dielectric in a Time-Varying Field

Polarization current and displacement current. A change in polarization leads to a motion of charges. Indeed, as follows from the definition of polarization,

$$\mathbf{P} = <q\mathbf{l}>$$

A change in the moment of dipoles, $q\mathbf{l}$, can always be represented as a change in distance $\mathbf{l}$ between charges $\pm q$ which remain invariable. In this case

$$\partial \mathbf{P}/\partial t = <q\partial \mathbf{l}/\partial t> = <q\mathbf{v}> \tag{3.31}$$

where, as earlier, the symbol $< >$ denotes averaging over the volume referred to a unit volume.

The right side of (3.31) expresses the mean velocity of bound charges (referred to a unit volume), i.e. the density of current due to the motion of bound charges with changes in polarization; it is called the *density of polarization current*:

$$\mathbf{J}_P = \partial \mathbf{P}/\partial t \tag{3.32}$$

We can conclude from the above that the right side of Maxwell's first equation must contain a displacement current in vacuum (the term is quite conditional and anachronistic) along with the density of current due to transport of free charges, $\mathbf{J}$, and polarization current density:

$$\operatorname{curl} \mathbf{H} = \mathbf{J} + \partial \mathbf{P}/\partial t + \varepsilon_0 \partial \mathbf{E}/\partial t \tag{3.33a}$$

Using the vector $\mathbf{D}$, we can convert (3.33a) to an expression of the form

$$\operatorname{curl} \mathbf{H} = \mathbf{J} + \partial \mathbf{D}/\partial t = \mathbf{J}_t \tag{3.33b}$$

where the right side expresses, as before, the total current density that also includes the density of polarization current. The term $\partial \mathbf{D}/\partial t$ is called *the density of displacement current.*

Applying (3.33) to harmonic (periodic) phenomena, we can replace all the quantities by complex counterparts, and the derivatives $\partial/\partial t$ by the factor $j\omega$, as done in (2.8) and (2.9). The two summands of the displacement current density, $\partial \mathbf{P}/\partial t$ and $\varepsilon_0 \partial \mathbf{E}/\partial t$, are given different interpretations. The first results from changes in the state of a substance and constitutes the flow of charges, though they are just bound charges. The second reflects only one of the properties of an electromagnetic field itself, i.e. a time-varying electric field that is the curl of a magnetic field and can be accompanied by a magnetic field.

Polarization delay. Where polarization of a medium is due to the displacement of electron orbits (nonpolar molecules), the dipole moment builds up practically at an instant, i.e. immediately after the application of an electric field. In the case of polar molecules, however, the polarization occurs as a result of rotation of the molecules, and so the mean dipole moment sets in with a certain delay, which varies with the types of dielectric.

It is natural that the permittivity of dielectrics decreases with increasing frequencies because of a delay in polarization. Thus in most high-permittivity magnetodielectrics the value of ε rapidly decreases with a rise in frequencies and does not exceed 12 in the superhigh frequency (SHF) band ($f \approx 10^9$ to 10^{10} Hz).

Dielectrics with a higher permittivity generally exhibit a greater lag in polarization, though there are some exceptions. For example, the permittivity of rutile keeps high, over 100, even in the SHF band.

In the simplest case, the delay in polarization of a linear medium is described by the equation [1] in differential form

$$P + \tau \partial P/\partial t = \varepsilon_0 k_e E \tag{3.34}$$

From this equation it is apparent at once that here as before the factor k_e determines the polarization under the steady-state conditions and in a stationary field ($\partial P/\partial t = 0$). But at $\partial P/\partial t > 0$, i.e. with the polarization process gathering way, P is lower than its steady-state value:

$$P = \varepsilon_0 k_e E - \tau \partial P/\partial t = \varepsilon_0 k_e (E - \alpha \partial P/\partial t) \tag{3.35}$$

The expression within parentheses may be regarded as a field weakened by friction, $\alpha \partial P/\partial t$, or as an effective (actual) field.

The equations similar to that given above are often met in physics.

It is useful to compare such an equation as (3.34) and the equation for a series RC circuit:

$$u_C + \tau du_C/dt = u$$

where $\tau = RC$ is the time constant.

Of interest is also the comparison of (3.34) and the classical equation of mechanics

$$x + \tau dx/dt = kf \tag{3.36}$$

which describes elastic deformation, for example, tension of a spring in the presence of friction proportional to the rate of motion. This friction is called viscous or newtonian. The effect of inertial forces proportional to acceleration, dx^2/dt^2, is not taken into account here. These forces may be regarded to be included into the right side of the equation.

Equation (3.35) is quite easy to solve. For example, under zero initial conditions and with the step-like changes in $E = E_0 \cdot 1(t)$,

$$P = \varepsilon_0 k_e E_0 (1 - e^{-t/\tau}) \tag{3.37}$$

For steady-state periodically varying conditions, it is convenient to represent in complex form both equation (3.35)

$$\dot{P} + \tau j\omega \dot{P} = \varepsilon_0 k_e \dot{E} \tag{3.38}$$

and its solution

$$\dot{P} = \varepsilon_0 [k_e/(1 + j\omega\tau)] \dot{E} \tag{3.39}$$

For the case under study, the electric flux density in complex notation is obviously given by

$$\dot{D} = \varepsilon_0 \dot{E} + \dot{P} = \varepsilon_0 [(1 + k_e + j\omega\tau)/(1 + j\omega\tau)] \dot{E} \tag{3.40}$$

---

[1] In words, this equation states that the rate of change of polarization, $\partial \mathbf{P}/\partial t$, is proportional to the difference between the steady-state polarization $P_\infty = \varepsilon_0 k_e E$ and the polarization P actually established at a given moment.

and at $\omega\tau \ll k_e \gg 1$, the electric flux density is expressed by the approximate equation of the form

$$\dot{D} = \varepsilon_0[(1 + k_e)/(1 + j\omega\tau)]\,\dot{E} \tag{3.41}$$

But at high values of $\omega\tau$ and, in particular, at $\omega\tau \to \infty$, it is only the full expression (3.40) that gives a physically correct result:

$$\dot{D} \to \varepsilon_0 \dot{E} \tag{3.42}$$

Complex representation of susceptibility and permittivity. Leaving the statements of susceptibility and permittivity for ac fields intact, we must represent these factors in complex form, as is clear, for example, from the latter equations. If there is a need to stress the complex nature of these factors, a bar is drawn under the respective symbols:

$$\underline{\varepsilon} = \varepsilon' - j\varepsilon''; \quad \underline{k} = k' - jk'' \tag{3.43}$$

where the prime and the double prime denote respectively the real and the imaginary part of complex quantities.

Assuming $\dot{P} = \varepsilon_0 \underline{k}\,\dot{E}$, we find that for (3.39)

$$\underline{k} = k' - jk'' = k_e/[1 + (\omega\tau)^2] - j\{k_e\omega\tau/[1 + (\omega\tau)^2]\} \tag{3.44}$$

where $k_e = \underline{k}(0) = k(0)$ is the susceptibility at $\omega = 0$.

In a similar manner, setting $\dot{D} = \varepsilon_0 \underline{\varepsilon}\,\dot{E}$, we find that for (3.40)

$$\begin{aligned}\underline{\varepsilon} = \varepsilon' - j\varepsilon'' &= [\varepsilon(0) + (\omega\tau)^2]/[1 + (\omega\tau)^2] \\ &\quad - j\{[\varepsilon(0) - 1]\omega\tau/[1 + (\omega\tau)^2]\}\end{aligned} \tag{3.45}$$

where $\varepsilon(0) = 1 + k_e$ is the permittivity at $\omega \to 0$.

As seen from the derived expressions, both susceptibility and permittivity may depend on frequency.

The frequency response of susceptibility may be rather complex in character. But in most cases the frequency response reduces precisely to the dependence specified by (3.44). The factor τ is the time constant, or relaxation time. The term *relaxation* refers to all differential equations given above and also to their solutions including the frequency characteristic.

The hodograph, i.e. the locus of quantity $\underline{k}(j\omega)$ in the complex plane, is a circle diagram. Whatever the frequency dependence $\underline{\varepsilon}(j\omega)$, the relation between $\varepsilon'(j\omega)$ and $\varepsilon''(j\omega)$ is quite definite if the dielectric remains linear. This relation is seen from the Kramers-Kronig permittivity dispersion expressions:

$$\varepsilon'(\omega_1) = 2/\pi \int_0^\infty \{\omega\varepsilon''(\omega)/[\omega^2 - \omega_1^2]\}d\omega + \varepsilon'(\infty)$$

$$\varepsilon''(\omega_1) = -(2/\pi)\omega_1 \int_0^\infty \{\varepsilon'(\omega)/[\omega^2 - \omega_1^2]\}d\omega$$

where ω_1 is the frequency at which it is necessary to define permittivity.

Thus, the real part of $\underline{\varepsilon}$ cannot be constant if the imaginary part is different from zero: ε' is constant only if $\varepsilon'' = 0$. Notice that the dependences in question are similar to the frequency characteristics for two-terminal networks. Where the experimental results do not follow the Kramers-Kronig relationships, it can be stated

with confidence that these results are in error if, of course, the measurements are made on linear dielectrics.

In many dielectrics, the frequency characteristic contains not one but two time constants τ_1 and τ_2, or even more.

Reactive and active components of displacement current. In the absence of polarization delay, the displacement current leads the field strength in phase by exactly $\pi/2$:

$$\dot{J} = j\omega\varepsilon_0\varepsilon\,\dot{E} \tag{3.46}$$

The displacement current is in quadrature with the field strength and hence can be called reactive. This is only the case if there is no delay in the processes of polarization. But if the polarization cannot keep up with changes in the field, the displacement current leads the field strength by less than $\pi/2$, which points to the appearance of an active displacement current component that is in phase with the field strength. The active component adds to the density of conduction current.

The complex permittivity then has a negative imaginary part. Indeed, if the permittivity is $\underline{\varepsilon} = \varepsilon' - j\varepsilon''$, the displacement current density takes the form

$$\dot{J} = j\omega\varepsilon_0\underline{\varepsilon}\,\dot{E} = (j\omega\varepsilon_0\varepsilon' + \omega\varepsilon_0\varepsilon'')\dot{E} \tag{3.47}$$

The addend in the parantheses expresses the displacement current component

$$\dot{J}_a = \omega\varepsilon_0\varepsilon''\dot{E} \tag{3.48}$$

which is in phase with the field strength and can be called active. Its appearance is attributed to power dissipation in accordance with Joule's law

$$P_0 = J_a E = \omega\varepsilon_0\varepsilon'' E^2 \tag{3.49}$$

The ratio $\varepsilon''/\varepsilon'$ is often called the *dissipation factor*, or *loss tangent*

$$\tan\delta = \varepsilon''/\varepsilon' \tag{3.50}$$

By formulas (3.49) and (3.50),

$$P_0 = \omega\varepsilon_0\varepsilon' E^2 \tan\delta \tag{3.51}$$

At low values of $\tan\delta$, one can assume to a sufficient degree of accuracy that $\varepsilon = |\underline{\varepsilon}| \approx \varepsilon'$ (the error is less than 1 % at $\tan\delta$ below 0.1). Instead of ε', we then may simply write ε and, hence,

$$P_0 = \omega\varepsilon_0\varepsilon E^2 \tan\delta \tag{3.52}$$

This formula is applicable to good dielectrics in which $\tan\delta$ comes to hundredths, thousandths, and even ten thousandths of unity.

Dielectric loss. The power P_0 is dissipated in a dielectric even if it does not display conduction. It is called *power loss*. In accordance with (3.35), this power may be interpreted as the loss due to friction with changes in polarization. The dielectric loss results in heating of a dielectric, which is commonly undesirable (in capacitors, cables, insulators). But high-frequency heating of dielectrics finds practical applications, for example, in the manufacture of plastics, for rubber heating, in food industry, in glassmaking, etc.

Imperfect dielectrics and poor conductors. All dielectrics are imperfect insulators to a greater or lesser degree. They have a finite, though very low, conductivity ($\sigma \neq 0$). In an alternating field, therefore [see Eqs. (3.33) and (2.9)], the total current is a sum of the conduction current, $J = \sigma E$, and displacement current. For a steadily varying field, Maxwell's first equation in complex form is then given by

$$\operatorname{curl}\dot{\mathbf{H}} = (\sigma + j\omega\varepsilon_0\underline{\varepsilon}\,)\,\dot{\mathbf{E}} = [\sigma + j\omega\varepsilon_0(\varepsilon' - j\varepsilon'')]\,\dot{\mathbf{E}} \tag{3.53a}$$

or

$$\operatorname{curl}\dot{\mathbf{H}} = [\sigma + \omega\varepsilon_0\varepsilon'' + j\omega\varepsilon_0\varepsilon']\dot{\mathbf{E}} \tag{3.53b}$$

In the latter equality, the complex permittivity is presented in terms of the real part, ε', and imaginary part, ε'', as given by (3.43). The equation contains only a real quantity σ, which is always true up to very high radio frequencies.

The electric properties of a substance subjected to an ac field of a given frequency are described by all the terms inside the brackets of (3.53). The imaginary part of permittivity, ε'', is responsible for the current component in phase with the conduction current. Therefore, it is practically impossible to distinguish σ from $\omega\varepsilon_0\varepsilon''$ by the results of an experiment conducted in an ac field of constant frequency.

Equivalent conductivity and equivalent permittivity. The complete multiplier at $\dot{\mathbf{E}}$ on the right side of Maxwell's first equation, i.e. all the terms in the brackets of (3.53), can be expressed as one complex factor by introducing an equivalent complex quantity $\underline{\sigma}_{eq}$ for a conducting medium:

$$\operatorname{curl}\dot{\mathbf{H}} = \underline{\sigma}_{eq}\,\dot{\mathbf{E}} \tag{3.54}$$

or by introducing an equivalent complex permittivity for a dielectric medium:

$$\operatorname{curl}\dot{\mathbf{H}} = j\omega\varepsilon_0\underline{\varepsilon}_{eq}\dot{\mathbf{E}} \tag{3.55}$$

Comparing (3.53) with (3.54) and (3.55), we can readily find that

$$\underline{\sigma}_{eq} = \sigma_1 + j\sigma_2 = \sigma + \omega\varepsilon_0\varepsilon'' + j\omega\varepsilon_0\varepsilon' \tag{3.56}$$

and that

$$\underline{\varepsilon}_{eq} = \varepsilon_1 - j\varepsilon_2 = \varepsilon' - j(\varepsilon'' + \sigma/\varepsilon_0\omega) \tag{3.57}$$

The equivalent imaginary part of permittivity contains here a component proportional to conductivity:

$$\varepsilon_2 = \varepsilon'' + \sigma/\varepsilon_0\omega \tag{3.58}$$

With a decrease in frequency ($\omega \to 0$), the equivalent imaginary part of permittivity infinitely grows.

It is apparent from the two latter equations that the total loss tangent for a medium exhibiting $\underline{\varepsilon}_{eq}$ is

$$\tan\delta_{eq} = \varepsilon_2/\varepsilon_1 = [\underline{\sigma} + \varepsilon_0\omega\varepsilon'']/(\varepsilon_0\varepsilon'\omega) \tag{3.59}$$

The total power loss may then be expressed as

$$P_0 = \omega\varepsilon_0\varepsilon_2 E^2 = \omega\varepsilon_0\varepsilon_1 \tan\delta_{eq} E^2 \tag{3.60}$$

Boundary conditions at the interface between two imperfect dielectrics. The condition for the normal components of an electric field strength becomes more com-

plicated here because there is a need for considering both complex permittivity and complex quantity $\underline{\sigma}_{eq}$.

From Maxwell's first equation,

$$\operatorname{curl} \dot{\mathbf{H}} = j\omega\varepsilon_0 \underline{\varepsilon}_{eq} \dot{\mathbf{E}}$$

it is obvious that the vector $\underline{\varepsilon}_{eq}\dot{\mathbf{E}}$ is source-free since the divergence of any curl is identically equal to zero. Therefore, at the interface between two imperfect dielectrics a and b,

$$\underline{\varepsilon}_{eq,\,a}\dot{E}_{a,\,n} = \underline{\varepsilon}_{eq,\,b}\dot{E}_{b,\,n} \tag{3.61}$$

Similarly,

$$\underline{\sigma}_{eq,\,a}\dot{E}_{a,\,n} = \underline{\sigma}_{eq,\,b}\dot{E}_{b,\,n} \tag{3.62}$$

since $\underline{\sigma}_{eq}$ differs from $\underline{\varepsilon}_{eq}$ only by the constant factor $j\omega\varepsilon_0$.

For tangential components at the interface between two media, the condition derived earlier remains valid:

$$\dot{E}_{a,\,t} = \dot{E}_{b,\,t} \tag{3.63}$$

Expressions (3.61) through (3.63) are suitable for use in all the calculation and analysis methods developed for ideal dielectrics with the real part of permittivity.

As seen from expressions (3.61) and (3.63), the field strength $\dot{E}_a$ is generally out of phase with $\dot{E}_b$.

Free and bound charges expressed in complex form. When there is a possibility of deriving the conductivity σ from the complex parameter of a medium, it is natural to determine a free charge using (2.29) written in complex notation:

$$\operatorname{div} \dot{\mathbf{J}} = \operatorname{div}(\sigma\dot{\mathbf{E}}) = -j\omega\dot{\rho} \tag{3.64}$$

Here ρ and $\dot{\mathbf{J}}$ are the complex quantities of the volume density of a free charge density and the conduction current density.

The surface density of a free charge at the interface between two media is therefore written as

$$j\omega\dot{\xi} = \dot{J}_{a,\,n} - \dot{J}_{b,\,n} = \sigma_a\dot{E}_{a,\,n} - \sigma_b\dot{E}_{b,\,n} \tag{3.65}$$

where the positive direction for the normal component of conduction current density, $\dot{J}_{a,\,n}$, corresponds to the flow of charges to the interface; the contrary is true for the normal component $\dot{J}_{b,\,n}$.

Substituting into (3.65) the expressions for σ_a and σ_b from equalities

$$\underline{\sigma}_a = \underline{\sigma}_{eq,\,a} - j\omega\varepsilon_0\varepsilon_a$$

$$\underline{\sigma}_b = \underline{\sigma}_{eq,\,b} - j\omega\varepsilon_0\varepsilon_b$$

set up in accordance with (3.56) and taking into account (3.62), we find that Gauss' theorem in complex form holds for imperfect dielectrics:

$$\dot{\xi} = \varepsilon_0(\varepsilon_b\dot{E}_{b,\,n} - \varepsilon_a\dot{E}_{a,\,n}) = \dot{D}_{b,\,n} - \dot{D}_{a,\,n} \tag{3.66}$$

where the normal is in the direction from medium a to medium b.

Example 3.1. In a parallel-plate capacitor with two-layer insulation (Fig. 3.10), $\varepsilon_a = 7$ in layer $a = 0.05$ cm and $\varepsilon_b = 2$ in layer $b = 0.20$ cm. The conductivity of both layers is the

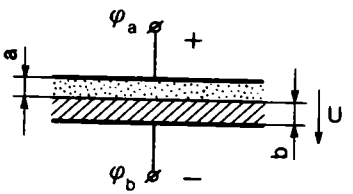


Fig. 3.10

same, $\sigma = 1.5 \times 10^{-10}$ S/cm. The capacitor is connected to a source of voltage $\dot{U} = 2.5$ kV at a frequency: (A) $f = 50$ Hz; (B) $f = 5$ kHz; (C) $f = 0$.

Problem: determine (1) the field strength in layers a and b and (2) free charge density ξ at the dielectric-dielectric boundary.

Solution. 1. From Eqs. (3.61) and (1.9), namely, from

$$\underline{\varepsilon}_{eq,a}\dot{E}_a = \underline{\varepsilon}_{eq,b}\dot{E}_b$$

and

$$a\dot{E}_a + b\dot{E}_b = \dot{U} \tag{a}$$

where

$$\underline{\varepsilon}_{a,b} = \varepsilon_{a,b} + \sigma/j\omega\varepsilon_0 \tag{b}$$

we determine (in kV/cm):

$$\text{(A) } \dot{E}_a = 7.12\angle -27°40', \quad \dot{E}_b = 10.9\angle 4°20'$$

$$\text{(B) } \dot{E}_a = 3.33, \quad \dot{E}_b = 11.65$$

In the latter (B) case, the field distribution practically coincides with the distribution when $\sigma \approx 0$ and $\underline{\varepsilon}_a = \varepsilon_a$, $\underline{\varepsilon}_b = \varepsilon_b$; (C) $E_a = E_b = 10$ kV/cm at $\omega \to 0$.

2. Knowing $\dot{E}_{a,b}$ and σ, we determine from (3.65) the free charge density at the dielectric-dielectric boundary (in C/cm^2): (A) $\dot{\xi} = (-1.98 + j2.20)10^{-9}$; (B) $\dot{\xi} = j3.97 \times 10^{-11}$; (C) $\xi = -4.43 \times 10^{-9}$ at $\omega \to 0$.

Check. It is interesting to verify the obtained results for the surface density of a free charge by use of formula (3.66), namely,

$$\dot{\xi} = \dot{D}_{b,n} - \dot{D}_{a,n} = \varepsilon_0(\varepsilon_b\dot{E}_b - \varepsilon_a\dot{E}_a) \tag{c}$$

For (A), i.e. at $f = 50$ Hz, we get $\dot{\xi} = (-1.97 + j2.20)10^{-9}$ C/cm^2, and for (B), i.e. at $f = 5$ kHz,

$$\underline{\varepsilon}_a = \varepsilon_a + \sigma/j\omega\varepsilon_0 = 7 - j0.054$$
$$\underline{\varepsilon}_b = 2 - j0.054 \tag{d}$$

One cannot simply delete small imaginary components here but must continue to perform algebraic transformations. However, it is first necessary to make sure that the difference between the values of quantities, being very close in value but yet different from zero, is quite definite.

Using formulas (a) and (b), we can readily determine $\dot{E}_a$ and $\dot{E}_b$ in general form. Substituting these expressions into (c) gives

$$\dot{\xi} = \varepsilon_a[\dot{U}/(b\underline{\varepsilon}_a + a\underline{\varepsilon}_b)](\varepsilon_b\underline{\varepsilon}_a - \varepsilon_a\underline{\varepsilon}_b) \tag{e}$$

In the denominator of (e) we may set $\underline{\varepsilon}_a \approx \varepsilon_a$ and $\underline{\varepsilon}_b \approx \varepsilon_b$. However, such an approximation is impermissible for the expression $(\varepsilon_b\underline{\varepsilon}_a - \varepsilon_a\underline{\varepsilon}_b)$. Substituting the equivalent complex permittivities defined in (d), we obtain

$$\varepsilon_b\underline{\varepsilon}_a - \varepsilon_a\underline{\varepsilon}_b = \sigma(\varepsilon_b - \varepsilon_a)/j\omega\varepsilon_0$$

Finally, from (e) we find that $\dot{\xi} = j3.97 \times 10^{-11}$ C/cm^2, which is in argeement with the result determined by use of (3.65).

Is it a conductor or dielectric? As seen from (3.56), the equivalent complex quantity $\underline{\sigma}_{eq}$ is frequency dependent. The same is true for $\underline{\varepsilon}_{eq}$ given by (3.57).

If we assume $\varepsilon'' = 0$ and $\varepsilon' =$ constant (or even $\varepsilon'' =$ constant $\neq 0$ and $\varepsilon' =$ constant over a certain frequency range), it will be apparent that the reactive component of $\underline{\sigma}_{eq}$ grows with frequency. This means that displacement currents rise in magnitude as compared to conduction currents. At low frequencies, a medium may sometimes be regarded as a conductor:

$$\sigma + \omega\varepsilon_0\varepsilon'' \gg \omega\varepsilon_0\varepsilon' \tag{3.67}$$

and at high frequencies, it may be regarded as a dielectric if

$$\omega\varepsilon_0\varepsilon' \gg \sigma + \omega\varepsilon_0\varepsilon'' = \omega\varepsilon_0\varepsilon_2 \tag{3.68}$$

In electrical engineering and primarily in high-voltage applications, it is sometimes important to know the field distribution throughout the insulation. As seen from (3.56), the dielectric properties of insulation do not play any role in the distribution of field at frequencies approaching zero since in this case $\underline{\sigma}_{eq} \approx \sigma$: it is only the conductivity that determines the dc field distribution. At high frequencies, however, the field distribution throughout the insulation can depend on its dielectric properties (see Example 3.1).

The permittivity of many media is frequency dependent, i.e. $\varepsilon'' = \varepsilon''(\omega)$ and $\varepsilon' = \varepsilon'(\omega)$. The effects observed in this case show up as follows. In a low-frequency field, such materials as glass and vanish-impregnated paper behave as perfect dielectrics showing very low σ and rather small $\tan\delta$. But as the frequency rises, ε' drops sharply, ε'' and hence $\tan\delta$ begin to grow, and so these materials cease to function as insulators and heat up heavily in strong fields. The ionosphere enveloping the Earth proves a good conductor and does not let pass through itself both low-frequency and very high-frequency radio waves. That is why only definite frequency bands are suitable for communication with space vehicles and for radio astronomic observations.

Frequency characteristics and transient processes. The extent of information given by the frequency response of a linear circuit is well known. The characteristic $\underline{\varepsilon}_{eq}(j\omega)$ or $\underline{\sigma}_{eq}(j\omega)$ gives the same amount of information useful for field calculations.

In the calculation of sinusoidally varying (ac) fields, account is taken of $\underline{\varepsilon}_{eq}$ or $\underline{\sigma}_{eq}$ at specified frequencies. The things are different with aperiodic disturbances. Indeed, an aperiodic disturbance contains a spectrum of frequencies, and the frequency-dependent parameters of a medium respond differently to different frequencies.

In the calculation of a transient process, it is very convenient to use the operator form of the equivalent parameters of a medium, replacing $j\omega$ by p in expressions for $\underline{\varepsilon}_{eq}$ or $\underline{\sigma}_{eq}$. The operator expressions for equivalent permittivity and conductivity then become:

$$\begin{aligned} \varepsilon_{eq}(p) &= \varepsilon(p) + \sigma/p\varepsilon_0 \\ \sigma_{eq}(p) &= \sigma + p\varepsilon_0\varepsilon(p) \end{aligned} \tag{3.69}$$

These formulas are written on the assumption that the conductivity σ does not depend on frequency, while the permittivity is frequency dependent; it should be noted

that the permittivity in question is precisely $\underline{\varepsilon} = \varepsilon' - j\varepsilon''$ and not only the equivalent permittivity. In this case, the permittivity is given in operator form, $\varepsilon(p)$. For the relaxation-type frequency dependence (3.45), the permeability in operator form is then

$$\varepsilon(p) = [\varepsilon(0) + p\tau]/(1 + p\tau) \tag{3.70}$$

It should be remembered for sure that we can pass to operator notation only if we proceed from the full function of the complex argument $j\omega$ by substituting p for $j\omega$. That is why there cannot be expressions in operator form for ε' or ε'' numerically specified for any frequency or frequency range. As a matter of fact, an external influence must be expressed in operator form too. The solution assumes the simplest form under zero initial conditions. The easiest way for the solution in the nonzero initial conditions is to apply the superposition principle.

Frequency response is of large importance not only for the calculation of a field in a given medium, but also for the study of the physicochemical properties and structure of substances.

Capacitor in an electric circuit that shows a given change in polarization. Consider a parallel-plate capacitor connected to an external circuit (Fig. 3.11). For simplicity of calculations, we regard the external circuit as an ordinary resistor R. The general methods of calculation of transients enable us to perform further generalizations quite easily, assuming the capacitor to be connected across the impedance $Z(p)$ in operator form.

Let us place between the capacitor plates a strip whose polarization changes instantly to P_0 and remains constant. It may be assumed that the polarization arises from a mechanical pressure instantly applied to the strip.

The strip and plates have the same area, S. The strip thickness is a and the air gap between one of the plates and the strip is b. Determine the voltage across the capacitor, $u_C(t)$.

The capacitor stays uncharged at $t < 0$ ($u_C = 0, P = 0$). This initial condition is sufficient if the capacitor is connected to a resistor[1].

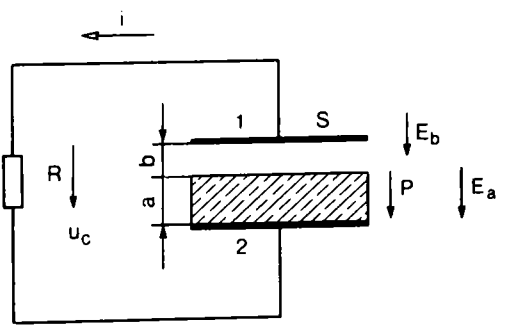


Fig. 3.11

[1] See the theorem for an induced charge, given in Sec. 4.8, and, in particular, Example 4.5.

Set up the following equations for the system under examination:

$$u_C = \varphi_1 - \varphi_2 = iR = E_a a + E_b b \tag{3.71a}$$

and

$$D_a = \varepsilon_0 E_a + P_0 = D_b = \varepsilon_0 E_b = \xi \tag{3.71b}$$

The electric flux density is the same both in air and in the strip because there is no free charge at the air-strip boundary; this flux density is equal to the surface charge density on the inside of plate *1*. The current in the circuit is equal to the rate of decrease of the charge on the upper plate:

$$i = -S\partial\xi/\partial t \tag{3.72}$$

which directly follows from the principle of conservation of charge. This current completes the circuit as a displacement current:

$$i = -S\partial D_b/\partial t \tag{3.73}$$

Since $D_b = \xi$, both statements are identical.

Of much significance is to make a point of the fact that the formulation of all the above equations has certainly required choosing (arbitrarily) the positive directions of E, P, i, and u_C. How the choice has been made is clear from Fig. 3.11 and Eqs. (3.71) to (3.73).

On eliminating all the unknowns, excepting ξ, we come to the differential equation

$$d\xi/dt + [(a + b)/(\varepsilon_0 RS)]\xi = (a/\varepsilon_0 RS)P_0 \tag{3.74}$$

for which the initial value of ξ is obvious: $\xi(0) = 0$.

Indeed, the capacitor was not charged preliminarily, and so a change in the charge at the finite value of current over a vanishingly small time interval is equal to zero.

Solving the above equation, we have

$$\xi = [a(a + b)]P_0(1 - e^{-t/\tau}) \tag{3.75}$$

where $\tau = \varepsilon_0 RS/(a + b) = RC_0$, C_0 is the capacitance of the capacitor in the absence of polarization.

Knowing $\xi(t)$, it is an easy matter to determine the current, voltage, and also E_a and E_b.

It is of interest to note that the polarization and field strength within the polarized layer are in opposite directions, i.e. the field tends to depolarize the dielectric.

3.3. Stationary Magnetic Fields in Magnetized Media

Magnetic polarization — magnetization. As known long ago, it is not only moving charges that possess a magnetic field, but also magnetized bodies in which the flow of current in the common sense of the word is nonexistent. However, comparing the magnetic fields due to currents with those of magnetized bodies (permanent magnets), French physicist Ampère suggested that the magnetization of bodies should stem from microscopic currents existent in the bodies and forming microscopic closed loops.

With the ordered orientation of amperian currents, these form a magnetic moment which, on the average, is different from zero. It is this moment that accounts for the existence of a magnetic field despite the seeming absence of the ordered motion of charges and absence of the current flow through any cross section of a body. Such visualization of amperian currents is not always applicable, but then it is not at all necessary. What is only necessary to recognize as valid is that a moving charge and hence current exhibit a magnetic moment. Of great importance was a much later discovery that elementary particles possess a magnetic moment and angular momentum along with mass and charge. The former two vector quantities are known as *spins*. Apart from the four parameters mentioned above, quantum physics employs other quantities which play an important role in characterizing elementary particles. In determining the resultant macroscopic magnetization, it is of no significance to consider whether this magnetization results from the orientation of spins or orbital moments (i.e. the moments due to orbital motion of atomic electrons). On the contrary, a change in the mechanical moment that accompanies a change in magnetization depends on the nature of carriers of a magnetic moment, and therefore the attendant mechanical effects are different.

Magnetization is the magnetic polarization of a substance defined as the volume density of a magnetic moment

$$\mathbf{M} = \lim_{V \to 0} (\sum \mathbf{m}/V) \tag{3.76}$$

where $\mathbf{m}$ is the magnetic dipole moment.

The right side of (3.76) requires averaging such as done in defining the vector of electric polarization, $\mathbf{P}$.

From the definition itself it follows that the magnetic moment of an elemental volume dV of a magnetized body is

$$d\mathbf{m} = \mathbf{M}dV \tag{3.77}$$

The field of magnetic dipoles is defined by formulas (2.73) and (2.77). The vector potential for the magnetic field of a magnetized substance may be determined from (2.73) by summing up the potentials of all moments, $\mathbf{M}dV$. The vector potential is thus expressed as an integral

$$\mathbf{A}_M = \mu_0 \int (\mathbf{M} \times \mathbf{e}_R/4\pi R^2)\,dV \tag{3.78}$$

extended over the entire volume V where $M \neq 0$.

Density of coupled currents of a magnetized medium. Expression (3.78) lends itself to transformations largely similar to those performed in deriving equality (3.3) that relates $\mathbf{P}$ to φ_P. Note first of all that $\mathbf{e}_R/R^2 = \nabla(1/R)$, reasoning from all the assumptions built into (3.4). Further, using the formula

$$\operatorname{curl}(\mathbf{M}/R) = (1/R)\operatorname{curl}\mathbf{M} - \mathbf{M} \times \operatorname{grad}(1/R)$$

(see Appendix A2), we find that

$$\mathbf{M} \times \mathbf{e}_R/R^2 = \mathbf{M} \times \operatorname{grad}(1/R) = (1/R)\operatorname{curl}\mathbf{M} - \operatorname{curl}(\mathbf{M}/R) \tag{3.79}$$

Substituting (3.79) into (3.78) gives

$$\mathbf{A}_M = \mu_0 \int (\mathbf{M} \times \mathbf{e}_R/4\pi R^2)dV = \mu_0 \int (\operatorname{curl}\mathbf{M}/4\pi R)dV \tag{3.80}$$

The integral must be extended over the entire volume occupied by the magnetized medium. In the final expression we have omitted the integral of the second term given in (3.79) since it is equal to zero. Indeed, applying to this integral the modified Ostrogradsky-Gauss theorem which contains vector products instead of the scalar product of vectors (see formula 4, Appendix A3), we obtain

$$\int \operatorname{curl}(\mathbf{M}/R)dV = -\oint (\mathbf{M}/R) \times d\mathbf{S} \tag{3.81}$$

The integral must be taken over the entire volume occupied by the magnetized medium or even over a larger volume, which is permissible from the statement of the problem itself: the inclusion of excess volume elements in the integral of (3.80) adds nothing but zero. But the extension of integration beyond the boundary of region $M \neq 0$ reduces the integrals of (3.81) to zero because the closed surface S lies in the region $M = 0$.

Comparing (3.80) with (2.62a), i.e.

$$\mathbf{A}_M = \mu_0 \int (\operatorname{curl}\mathbf{M}/4\pi R)dV$$

$$\mathbf{A} = \mu_0 \int (\mathbf{J}/4\pi R)dV \tag{3.82}$$

allows us to state that the field of magnetized bodies is identical to the field of a spatially distributed current density equal to the curl of magnetization:

$$\mathbf{J}_M = \operatorname{curl}\mathbf{M} \tag{3.83}$$

This quantity is known as *the density of coupled currents.*

The transformations performed here presuppose that the vector $\mathbf{M}$ is continuous and differentiable. However, one has often to deal with surfaces of discontinuous magnetization. As noted earlier when considering polarization it is initially possible to assume that magnetization smoothly changes near the discontinuity surface and then to pass to the limit. Let the tangential component of magnetization change abruptly from $\mathbf{M}_0$ to zero on the surface of a body. Choose the cartesian coordinates so that the direction of vector $\mathbf{M}$ is along the x axis and the normal to the surface extends along the y axis. Also, suppose that magnetization varies linearly from $\mathbf{M}_0$ to zero in a thin layer of thickness b (Fig. 3.12) obeying the law

$$M = M_x = (M_0/b)(b - y) \text{ at } 0 \leqslant y \leqslant b \tag{3.84}$$

We thus assume that the magnetization near the observation point is independent of z.

Calculating the curl of (3.84) gives

$$\mathbf{J}_M = \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ M & 0 & 0 \end{vmatrix} = (-\partial M/\partial y)\mathbf{e}_z = \frac{M_0}{b}\mathbf{e}_z \tag{3.85}$$

The current density $\mathbf{J}_M$ is in the direction of the z axis and represents the vector product of magnetization by the normal to the discontinuity surface. The current

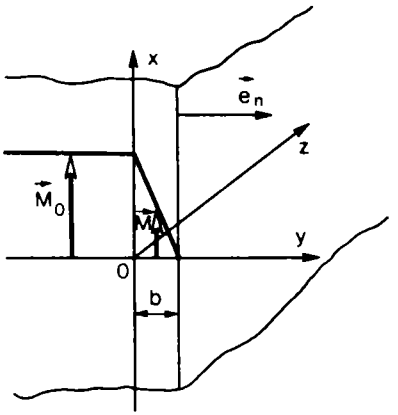


Fig. 3.12

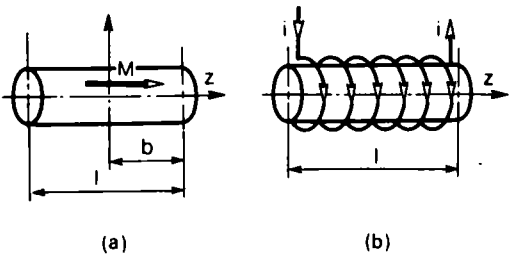


Fig. 3.13

density is constant and nonzero only over the width b; it may be given as the vector of surface current

$$\mathbf{K}_M = \mathbf{J}_M b = \mathbf{M}_0 \times \mathbf{e}_n \tag{3.86}$$

where $\mathbf{e}_n$ is the normal external to the discontinuity surface.

The found result does not depend on the chosen law of changes of $\mathbf{M}$ in a thin surface layer whose thickness may be taken to approach zero ($b \to 0$). If the magnetization at the discontinuity surface varies from $\mathbf{M}_1$ to $\mathbf{M}_2$, the equivalent density of surface current takes the form

$$\mathbf{K}_M = (\mathbf{M}_1 - \mathbf{M}_2) \times \mathbf{e}_n \tag{3.87}$$

which is not difficult to show by applying the method just described above; the normal $\mathbf{e}_n$ runs from medium *1* into medium *2*. The latter expression is known as the surface curl expression:

$$\text{Curl}\,\mathbf{M} = \mathbf{K}_M = (\mathbf{M}_1 - \mathbf{M}_2) \times \mathbf{e}_n \tag{3.88}$$

It replaces (3.83) at the surface of discontinuity.

Compare the magnetic field of a round core magnetized along its axis (Fig. 3.13*a*) and that of a solenoid of the same dimensions, having w_0 turns per unit length carrying current i (Fig. 3.13*b*). The magnetic fields of the core and solenoid must be the same if they have the same magnetic moments $\mathbf{m}$. But the total magnetic moment of the solenoid is

$$\mathbf{m} = iSw\mathbf{e}_z = iSw_0 l\mathbf{e}_z = iw_0 V\mathbf{e}_z \tag{3.89}$$

where V is the solenoid volume.

According to the definition, every turn has the same moment iS; $\mathbf{e}_z$ is the unit vector directed along the axis.

The magnetic moment of the core according to the definition of magnetization is given by

$$\mathbf{m} = \mathbf{M}V = MV\mathbf{e}_z \tag{3.90}$$

Hence, replacing the core magnetization by the equivalent surface current K_M, we must take

$$K_M = iw_0 = M \tag{3.91}$$

because the surface current density of the solenoid is just equal to iw_0.

Experience confirms that the magnetic fields of a magnetized core and a solenoid are equivalent if the core corresponds to the solenoid as stipulated above. One of the widespread methods of measuring magnetization lies exactly in the comparison of the magnetic effect of the magnetized body under examination and that of a solenoid whose geometric dimensions are known and the current is measured by an ammeter.

Example 3.2. The magnetization of a round cylinder along its axis (see Fig. 3.13*a*) is $M = 800$ A/cm, its length is $l = 2b = 6$ cm, and its diameter is $d = 2a = 3$ cm.

Problem. Find the B field on the z axis of the cylinder as a function of the distance from the middle of the magnet. Assume we know the calculation formula for the B field on the axis of a solenoid:

$$B = B_z = \mu_0 iw_0(\cos\alpha_1 - \cos\alpha_2)/2 \tag{a}$$

where iw_0 is the magnetomotive force (mmf) per unit length of the solenoid (per meter if μ_0 is expressed in H/m); α_1 and α_2 are angles formed between the solenoid z axis and radii drawn from the observation point to points on both end faces of the solenoid (Fig. 3.14); the angle α_1 corresponds to the end face in the direction of growth of z; the positive direction of current in the winding is in the direction of the z axis according to the right-hand screw rule.

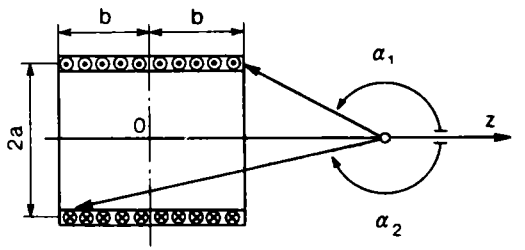


Fig. 3.14

Solution. Using formulas (3.86) through (3.91), we find the equivalent surface current for the magnetized core: $K_M = M = 800$ A/cm. This current is equivalent to the mmf per unit length of the solenoid having the same dimensions as the cylinder. So,

$$B = \mu_0 M(\cos\alpha_1 - \cos\alpha_2)/2$$

Counting off z from the middle of the magnet we get

$$\cos\alpha_1 = (b - z)/\sqrt{a^2 + (b - z)^2}$$
$$\cos\alpha_2 = -(b + z)/\sqrt{a^2 + (b + z)^2}$$

The B field is at a maximum in the middle of the magnet ($z = 0$):

$$B_{\max} = \mu_0 Mb/\sqrt{b^2 + a^2} = \mu_0 Ml/\sqrt{l^2 + d^2}$$

At the end faces of the magnet the flux density is $B_{end} = \mu_0 Mb/\sqrt{l^2 + a^2}$. If $4b^2 = l^2 \gg d^2$, the B_{end} is half as high as in the middle. For the parameters given in the example, $B_{\max} = 0.0900$ T (900 G) and $B_{end} = 0.0486$ T (486 G).

Magnetic field strength. Applying Maxwell's first equation to a magnetized medium,

$$\mathbf{J}_t = \mathbf{J} + \partial \mathbf{D}/\partial t \tag{3.92}$$

we should add to its right side the current density equivalent to magnetization and assume that

$$\operatorname{curl} \mathbf{B} = \mu_0(\mathbf{J}_t + \mathbf{J}_M) \tag{3.93}$$

The vector $\mathbf{J}_M$ entering this equation takes into account the effect of the magnetized medium on the magnetic field.

But the current density $\mathbf{J}_M$ is equal to the curl of magnetization (3.83), and hence

$$\operatorname{curl} \mathbf{B} = \mu_0(\mathbf{J}_t + \operatorname{curl} \mathbf{M}) \tag{3.94}$$

Transposing curl $\mathbf{M}$ to the left side and keeping in mind that the sum of derivatives is equal to the derivative of the sum, we obtain

$$\operatorname{curl}[(1/\mu_0)\mathbf{B} - \mathbf{M}] = \mathbf{J}_t \tag{3.95}$$

or, applying the integral notation (the circuital law), we have

$$\oint [(1/\mu_0)\mathbf{B} - \mathbf{M}]d\mathbf{l} = \int \mathbf{J}_t d\mathbf{S} \tag{3.96}$$

Expression (3.96) directly follows from (3.95) on integrating over the surface and using Stokes' theorem.

It is highly useful for the reader to see the similarity between the reasonings which have led to the two latter equations and the reasonings pertaining to an electrically polarized medium [see Sec. 3.1, Eqs. (3.11), (3.14), and (3.18)]. As we did in the case of an electric field, we can now make the following important step.

Using the vector difference $(\mathbf{B}/\mu_0) - \mathbf{M}$ (instead of vector $\mathbf{B}$), we can apply Maxwell's first equation (the circuital law) to magnetized media, taking into account in explicit form, as before, only the total current. This difference between the two vectors is referred to as the *magnetic field strength*

$$\mathbf{H} = (1/\mu_0)\mathbf{B} - \mathbf{M} \tag{3.97}$$

Employing this new quantity, for any medium we can write

$$\oint \mathbf{H}d\mathbf{l} = \int \mathbf{J}_t d\mathbf{S}, \text{ or } \operatorname{curl} \mathbf{H} = \mathbf{J}_t \tag{3.98}$$

Now the current density that represents the effect of magnetization is taken into account in implicit form by passing from the $\mathbf{B}$ field to the $\mathbf{H}$ field.

It is obvious from all the expressions that the unit for $\mathbf{H}$ in the SI system used here is the ampere per unit length (A/m).

In the Gaussian system, the magnetic field strength is defined by the equality

$$\mathbf{H} = \mathbf{B} - 4\pi\mathbf{M} \tag{3.99}$$

which certainly has the same meaning as (3.97). The units of measurement for $\mathbf{B}$, $\mathbf{H}$, and $\mathbf{M}$ are principally the same in the Gaussian system. To distinguish $\mathbf{B}$ from $\mathbf{H}$, however, one customarily calls the unit of $\mathbf{B}$ the gauss and the unit of $\mathbf{H}$ the oersted (Oe) though they are equal:

$$1\,\text{Oe} = (1/0.4\pi)\text{A/cm} \approx 0.8\,\text{A/cm} \tag{3.100}$$

Formulas (3.97) and (3.99) differ not only in units of measurement but also in

form. Therefore, the relations between **H** and **M** in these systems are different: the numerical value of **M** given in A/cm should be multiplied by 0.1 to obtain the value of **M** in gausses (and oersteds).

Example 3.3. A wire carrying direct current I_0 runs inside a ferromagnetic ring of rectangular cross section. The internal radius of the ring is r_0 and the external radius is $3r_0$. At the specified current, the magnetic field strength $H = H_0$ is sufficiently high and at a radius of $r \leqslant 1.5\ r_0$ the ferromagnet undergoes magnetization practically to saturation, $M = M_\alpha = M_s$.

Problem. Determine the magnetic flux density in the core at $r \leqslant 1.5\ r_0$ and find the equivalent current density in the ferromagnet for the given region.

Solution. It is apparent from the basic statements that in the region of interest,

$$B = B_\alpha = \mu_0(I_0/2\pi r + M_s)$$

According to (3.93) and by reference to Appendix A5,

$$\mathbf{J}_M = \operatorname{curl}(B/\mu_0) = (\mathbf{e}_z/r)[(\partial/\partial r)(I_0/2\pi + rM_s)] = \mathbf{e}_z M_s/r$$

Susceptibility and permeability. For materials of weak permeability and for weak fields, the intensity of magnetization is proportional to the magnetic flux density:

$$\mathbf{M} = c\mathbf{B} \tag{3.101a}$$

Obviously, **M** proves proportional to the magnetic field strength:

$$\mathbf{M} = k_m\mathbf{H} \tag{3.101b}$$

In turn, **B** must be in proportion to **H**:

$$\mathbf{B} = \mu_0(\mathbf{H} + \mathbf{M}) = \mu_0(k_m + 1)\mathbf{H} \tag{3.102}$$

or

$$\mathbf{B} = \mu_0\mu\mathbf{H} \tag{3.103}$$

The proportionality factor k_m relating **M** to **H** received the name magnetic *susceptibility*. The proportionality factor m relating **B** to **H** (multiplied by the magnetic constant μ_0) bears the name of *permeability*

$$\mu = 1 + k_m = B/\mu_0 H \tag{3.104a}$$

In the Gaussian system of units, the permeability has the same value, i.e.,

$$\mu = 1 + 4\pi\chi = B/H \tag{3.104b}$$

while the susceptibility χ is a factor of 4π smaller [see also Eq. (3.19) and the next equations that follow in the same section].

In SI, the product $\mu_0\mu$ known as the absolute permeability is in use sometimes:

$$\mu_a = \mu_0\mu$$

where μ is the relative permeability, also designated as μ_r.

The relation between B (or M) and H for ferromagnetic materials is intricate in character. In the B-H or M-H plane, this relation appears as an open or closed magnetization curve called a hysteresis (magnetic) loop. Fig. 3.15 shows various types of loop (the dashed line passing through the origin is that for $B/\mu_0 = M$). When the curve assumes the form of a loop, the notion of permeability as defined by equality (3.103) does not make sense, nor does the equality itself. It should be

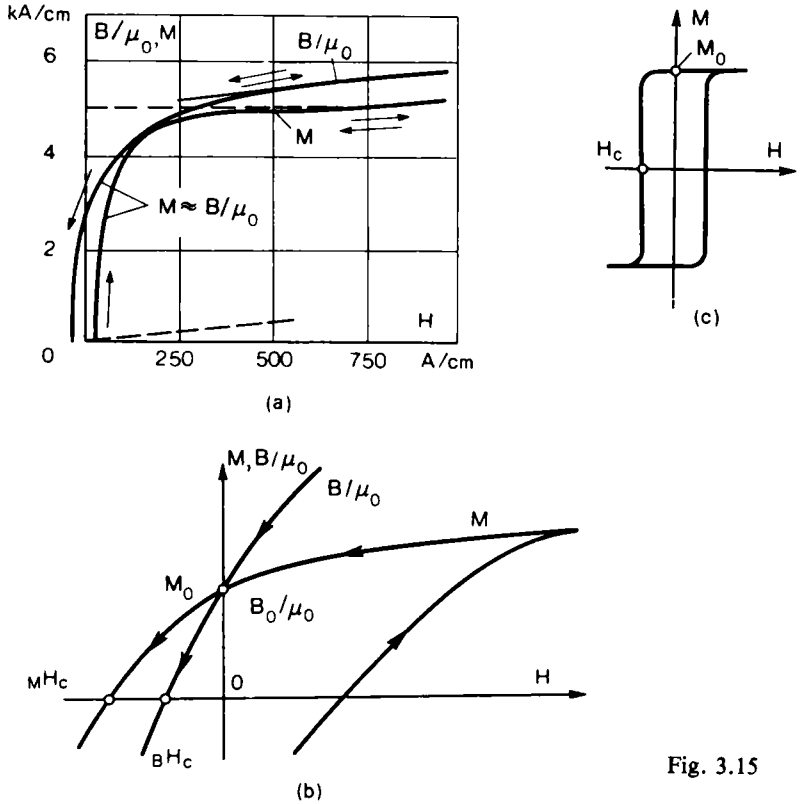


Fig. 3.15

pointed out that statement (3.97) as the basic definition of the concept of magnetic field strength proper always has meaning.

In electrical engineering calculations, the permeability of all materials, excepting ferromagnets, is commonly taken equal to unity. In SHF applications, resonance phenomena in ferrites heavily affect the permeability (see Fig. 3.21).

The strength of a field on the hysteresis loop at which $B = 0$ (or $M = 0$) is known as the induction coercive force ${}_BH_c$ (or magnetization coercive force ${}_MH_c$). At relatively low values of coercive force ($H_c/M_0 \ll 1$), ${}_BH_c$ is practically equal to ${}_MH_c \approx H_c$. The magnetic flux density and magnetization on the hysteresis loop at $H = 0$ are called the remanent induction (retentivity) and remanent magnetization, or just remanence (denoted by B_0 and M_0 or B_r and M_r respectively).

Careful consideration of the examples given below can help the reader to comprehend well the essence of the notions of B, M, and H.

Example 3.4. A current-carrying coil uniformly wound on a homogeneous toroidal core magnetizes the toroid. After cutting off the current, the remanent induction is found to be $B_0 = 1.6$ T.

Problem. Determine the remanent magnetization M_0 and make sure that formula (3.91) is valid.

Solution. In the absence of field strength[1], formula (3.97) gives

$$(1/\mu_0)B_0 = M_0 = 1.6/(4\pi 10^{-7}) = 1.27 \times 10^6 \text{ A/m}$$

Assuming that the field produced is due only to the surface current (in A/m) we apply the circuital law to B and get

$$B_0 = \mu_0 K_M \text{ or } K_M = M_0$$

Example 3.5. A coil uniformly wound on a circular core of ferromagnetic material (steel) carries a current magnetizing the core. On deenergizing the coil, the core retains magnetization M equal to 9.5 kA/cm or 950 G. The steel core has a small air gap of length $l_a = 0.02$ cm. The total length of the ferromagnetic core is $l_f = 10$ cm.

Problem. Find B and H inside the core and in the air gap assuming the B field at all points of the steel core and in the air gap to be the same ($B_a = B_f = B$) because the air gap is small.

Solution. According to the circuital law

$$H_f l_f + (1/\mu_0)Bl_a = iw = 0 \tag{a}$$

Substituting the expression for B, derived from the basic field-strength statement (3.97),

$$B = \mu_0(H_f + M) \tag{b}$$

into equation (a), we obtain

$$H_f(l_f + l_a) + Ml_a = 0 \tag{c}$$

or

$$H_f = -Ml_a/(l_a + l_f) \tag{d}$$

In the air gap

$$H_a = B/\mu_0 \tag{e}$$

with B determined from (b) at the given M and the found H_f:

$$B = \mu_0 Ml_f/(l_a + l_f)$$

The numerical values of quantities in the given example are as follows: $H_f = -19.0$ A/cm $= -23.8$ Oe, $B = 1.19$ T $= 11.19$ kG and $H_a = 9.48$ kA/cm $= 11.9$ kOe.

It is important to note that H_f is opposite in direction to M if the field is set up by the body of a given magnetization.

Example 3.6. What current must the core coil of Example 3.5 carry to reduce H_f of the core to zero after its magnetization? We may assume that the value of M here is invariable since the core is of a rectangular-loop material (Fig. 3.15*c*). What is H_a in the air gap in this case? What portion of B field can we ascribe to external current (B_i) and what portion can we attribute to substance magnetization (B_M)?

Solution. Unlike formula (a) of Example 3.5, the formula below set up from the circuital law has its right side different from zero:

$$H_f l_f + (1/\mu_0)Bl_a = iw \tag{a}$$

The statement of the problem is that H_f must be zero. So, $H_f = (B/\mu_0) - M = 0$ or

$$B = \mu_0 M \tag{b}$$

---

[1]Since $\oint \mathbf{H}d\mathbf{l} = H \cdot 2\pi r = iw = 0$, then $H = 0$. If the core is inhomogeneous, we cannot take H outside the integral and thus cannot assume that $H \doteq 0$ (see Example 3.5).

Consequently, from (a) and (b) we obtain

$$iw = Ml_a \tag{c}$$

The field strength in the air gap is then

$$H_a = B/\mu_0 = M$$

Here M is the magnetization of the ferromagnet as before.

The component due to magnetization must be the same as in the preceding example because the core geometry and magnetization have not changed:

$$B_M = \mu_0 M l_f/(l_f + l_a)$$

The component of B due to the current in the coil is

$$B_i = \mu_0 iw/(l_f + l_a)$$

Its value is the same as in the absence of ferromagnets.

The solution $iw = Ml_a$ implies that the current in the coil must make up for the lacking equivalent current (equivalent ampere-turns in the air gap) in order to achieve the conditions existing in a continuous uniformly magnetized toroid (see Example 3.4).

Example 3.7. A round ferromagnetic core (cylinder) has reached such a magnetization M that it may be considered constant and parallel to the axis (Fig. 3.16*a*). The cylinder radius is $a = 1.5$ cm, its length is $l = 2b = 6$ cm, and $M = 800$ A/m.

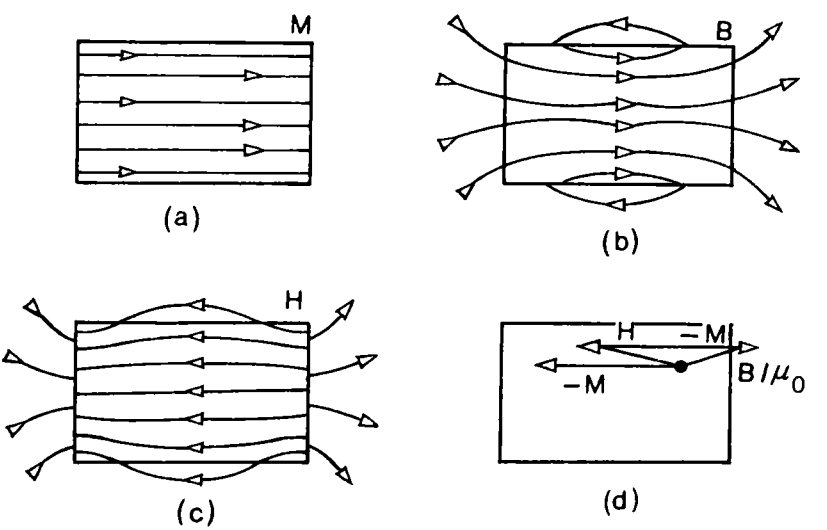


Fig. 3.16

Problem. Find the magnetic induction and field strength along the z axis coinciding with the axis of the cylinder.

Solution. The field on the axis of the cylinder in question is the same as given in Example 3.2. The H field is found as the difference between B/μ_0 and M. Inside the cylinder, on its axis

$$H = H_z = B/\mu_0 - M = M(\cos\alpha_1 - \cos\alpha_2)/2 - M$$

the angles $\alpha_{1,2}$ being shown in Fig. 3.14.

Inside the cylinder, the field strength on the axis is negative, i.e. its direction is opposite to the direction of M (see Example 3.5). Outside the cylinder, the field strength differs from B only in a constant factor the value of which depends on the choice of the system of units (μ_0 or 1).

It is of importance to note that the B normal component does not experience discontinuity on passing from the magnetized body to the environment, whereas the H normal component (normal to the end face) abruptly varies by a value of M.

Table 3.1 lists the values of B, H, and M in two systems of units for three values of z. The values are rounded off to a certain extent; $z = 0$ is the center of a core; for its end faces, $z = \pm b$; $\varepsilon \to 0$.

Table 3.1

z	M	B	H
0	800 A/cm (80 Oe)	9×10^{-2} T (900 G)	− 84 A/cm (− 106 Oe)
$b - \varepsilon$	800 A/cm (80 Oe)	4.86×10^{-2} T (486 G)	− 413 A/cm (− 520 Oe)
$b + \varepsilon$	0	4.86×10^{-2} T (486 G)	387 A/cm (486 Oe)

It is worthwhile to plot B, M, and H versus z and also to prove that

$$\int_{-\infty}^{+\infty} H_z dz = \oint \mathbf{H} d\mathbf{l} = 0$$

It is certainly possible to calculate B and H in other regions rather than on the axis only, though this involves substantial difficulties. In Fig. 3.16*b* and *c* are shown the **B** and **H** field lines inside and outside the core. A change in **B** inside the core is fully in accord with the change of **B** inside the coil of the same dimensions.

The vectors **B**, **H**, and **M** at off-axis points do not coincide in direction (Fig. 3.16*d*). But in this case too, $\mathbf{B} = \mu_0(\mathbf{H} + \mathbf{M})$.

Field vectors at the interface between two media and boundary conditions. From Gauss' theorem $\operatorname{div}\mathbf{B} = 0$ it follows that at the interface of two media the normal component of the B field remains continuous:

$$B_{2n} = B_{1n} \tag{3.105}$$

From the circuital law it is evident that the tangential components of field strength at any finite current density are continuous:

$$H_{2t} = H_{1t} \tag{3.106}$$

Indeed, to support the point, consider a rectangular loop (Fig. 3.17) whose two sides Δl are parallel to and lie on both sides of the interface and other two sides dn are normal to the interface and are very small even against the sides Δl. Applying the circuital law to this loop, we find that

$$\begin{aligned} \oint \mathbf{H} d\mathbf{l} &= (\mathbf{H}_1 + \mathbf{H}_2)\Delta\mathbf{l} = (H_{2t} - H_{1t})\Delta l \\ &= \mathbf{J}\Delta\mathbf{l} \times d\mathbf{n} = \mathbf{J}d\mathbf{S} \end{aligned} \tag{3.107}$$

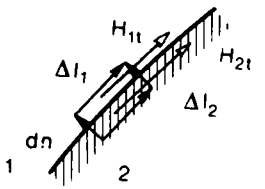


Fig. 3.17

or

$$H_{2t} - H_{1t} = J_\eta dn \to 0$$

where J_η is the component of current density that remains finite; this component is normal to the loop plane, i.e. parallel to $d\mathbf{S}$.

In analogy with (3.30), we can write at once that

$$\tan\beta_1/\tan\beta_2 = (H_{1t}/H_{1n})/(H_{2t}/H_{2n}) = H_{2n}/H_{1n} = \mu_1/\mu_2 \qquad (3.108)$$

Magnetostatics. We can extend the analogy still much further. In the absence of currents, i.e. when

$$\text{curl}\,\mathbf{H} = \mathbf{J} = 0 \qquad (3.109)$$

the magnetic field is potential, and so it is possible to specify the following system of similarities:

$$\begin{gathered} \mathbf{D} = \varepsilon\varepsilon_0\mathbf{E} \sim \mathbf{B} = \mu\mu_0\mathbf{H} \\ \mathbf{P} \sim \mu_0\mathbf{M} \\ \mathbf{E} = -\,\text{grad}\,\varphi \sim \mathbf{H} = -\,\text{grad}\;\varphi_M \end{gathered} \qquad (3.110)$$

A magnetic field differs from an electric field in that it has no free magnetic charges since div $\mathbf{B} = 0$. As for bound magnetic charges, these are defined by the expression

$$\text{div}(\mu_0\mathbf{H}) = -\,\text{div}(\mu_0\mathbf{M}) = \rho_M \qquad (3.111)$$

which is similar to (3.15).

Reasoning further from analogy, we may develop an expression for a scalar magnetic potential in terms of the distribution of volume charge density:

$$\varphi_M = 1/4\pi\mu_0 \int (\rho_M dV/R) \qquad (3.112)$$

where ρ_M is certainly defined by (3.111).

Magnetic phenomena are kinetic in character, though electric phenomena can be kinetic too. What mainly distinguishes the former phenomena from the latter is that magnetic monopoles do not exist in nature, at least only magnetic dipoles have so far been discovered. This is the reason why magnetostatics must deal with dipoles, i.e. magnetization. Some peculiarities become evident only in the calculation of dynamic interaction of the field with polarized bodies: in considering mechanical forces, one should assume that they act on the dipole as a whole, but not on bound charges being visualized as a dipole cut in two. The above reasoning, however, also refers to electrically polarized bodies immersed in an electric field.

Dipoles and the analog of Coulomb's experiment. Knowing the dynamic interaction between dipoles, we can determine their magnetic moment and field strength. This is the statement Gauss had made from his experiment. To a known extent, Gauss' experiment is the analog of Coulomb's experiment. Gauss' idea lies in that the oscillations of a dipole — a magnetic arrow having a moment $\mathbf{m}$ in a stationary field $\mathbf{H}$ — adhere to the equation

$$J\frac{\partial^2\alpha}{\partial t^2} + \mu_0 Hm\alpha = 0 \tag{3.113}$$

where α is the angle formed between the vectors $\mathbf{H}$ and $\mathbf{m}$; J is the moment of inertia of the arrow, assumed to be known; μ_0 is the magnetic constant, and t is time.

The solution of this equation for very weakly decaying oscillations takes the form

$$\alpha = A\sin\omega t \text{ at } \omega = 2\pi/T = \sqrt{\mu_2 Hm/J} \tag{3.114}$$

In the absence of an external field (the external field source being compensated for) and with the location of two identical arrows on the same axis, the field strength is determined from (1.25) and (2.76):

$$H = m/2\pi s^3 \tag{3.115}$$

where s is the distance between dipoles.

In this case, from (3.114) and (3.115) it follows that

$$m = \omega\sqrt{J2\pi s^3/\mu_0} \tag{3.116}$$

With the external field being present, the moment $\mathbf{m}$ can readily be evaluated by measuring the frequency twice: the first time, when the external H_0 field and the dipole H_d field are in the same direction, so that the resultant field $H = H_0 + H_d$, and the second time, when both dipoles turn through 180° and the resultant field is $H = H_0 - H_d$. Knowing the difference between the frequencies, we can calculate $\mathbf{m}$ and H_d.

In conclusion, let us point out that magnetization and polarization may be attributable to other sources and not only to the field strength. Where this is the case, $\mathbf{M}$ and $\mathbf{P}$ must be specified additionally one way or another. If $\mathbf{M}$ and $\mathbf{P}$ arise precisely from field vectors, the corresponding dependence, say, $\mathbf{P}(\mathbf{E})$ or $\mathbf{M}(\mathbf{H})$, must be given in addition to (3.17) and (3.97).

3.4. Ferromagnet in a Time-Varying Field

Main effects. A varying magnetic field gives rise to an electric field. An electric field in a conductor causes current. By Lenz' law, the induced current (eddy current) is always in such a direction as to oppose the change of the magnetic field. For ferromagnetic materials (the iron group) immersed in a time-varying field, this effect plays a dominant role. We shall discuss it in detail in Ch. 8. All the same, here we cannot but mention some of its features, for example, the demagnetizing action of eddy currents, which leads to a decrease in the mean permeability of ferromagnetic cores with increasing frequency. This certainly brings about thermal dissipation of the electromagnetic energy (Joule heat due to eddy currents).

The behavior of real ferromagnets in varying fields is very complex because these materials feature nonlinear response. The frequency response is then of no avail in judging the conditions at impulse input, which are highly important in engineering.

Complex susceptibility and complex permeability. The magnetization is not able to keep up with the changes in the field as is not the electric polarization. This effect in ac fields can be taken account of in the linear region by introducing complex quantities:

$$\underline{k} = k' - jk''$$
$$\underline{\mu} = \mu' - j\mu'' = 1 + \underline{k} \qquad (3.117)$$

If a change in magnetization depends on newtonian friction, i.e.

$$M + \tau dM/dt = k_m H \qquad (3.118)$$

then the susceptibility of a substance in a sinusoidally varying field is expressed as a complex quantity dependent on frequency:

$$\underline{k} = k' - jk'' = k_m/(1 + j\omega\tau) \qquad (3.119)$$

where $k_m = \underline{k}(0)$ is the susceptibility in a stationary field ($\omega \to 0$).

For the case where the field changes abruptly, $H(t) = H_0 \cdot 1(t)$, the solution to differential equation (3.118) gives us the known relaxation dependence

$$M(t) = k_m H_0 (1 - e^{-t/\tau}) \qquad (3.120)$$

Figure 3.18 illustrates the frequency response of the permeability of nickel-zink ferrite. This characteristic is in the main definable by (3.119). A drop in permeability is sometimes characterized by a few relaxation constants $\tau_1, \tau_2, \ldots$

At high frequencies, practically all ferromagnetic materials show a decay in permeability. The point is that we mean here the decay of permeability itself rather than the demagnetizing action of eddy currents. The assumption that internal friction indeed exists often allows us to explain well the phenomena being examined. In high-permeability materials, μ usually drops off at much lower frequencies. Because they have a high resistivity (millions of times that of metals), ferrites are practically free from the effect of eddy currents. All the same, the permeability of these materials noticeably decreases at 5 to 10 MHz if it was initially high (see Fig. 3.18).

Elliptic hysteresis loop. At a complex permeability

$$\underline{\mu} = \mu' - j\mu'' = \mu'(1 - j\tan\delta) \qquad (3.121)$$

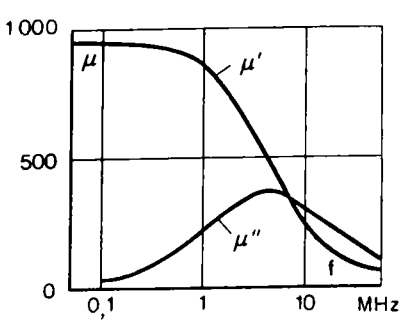


Fig. 3.18

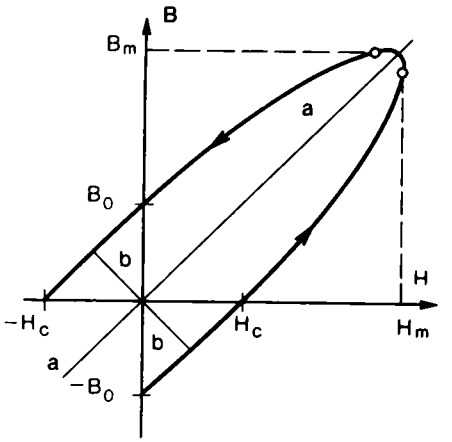


Fig. 3.19

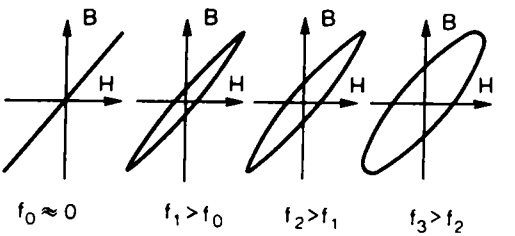


Fig. 3.20

the sinusoidally varying magnetic flux density

$$B(t) = B_m \sin(\omega t - \delta) \tag{3.122a}$$

lags the changes in magnetic field strength

$$H(t) = H_m \sin \omega t \tag{3.122b}$$

by an angle of $\delta = \arctan \mu'' / \mu'$.

It is easy to see that the last two formulas represent the parametric equation of an ellipse (Fig. 3.19).

As known and proved in Sec. 7.2, the energy dissipated in a unit volume per cycle is expressed as an area of the hysteresis loop (Warburg's law). The power dissipated per unit volume at $\mathbf{H} \parallel \mathbf{B}$ is then

$$P_0 = f \oint H dB \tag{3.123}$$

Substituting the expressions for B and H into (3.123), we find that the loop area per cycle is equal to

$$H_m B_m / 2 \int_0^{2\pi} \sin \omega t \cos(\omega t - \delta) d\omega t = \pi H_m B_m \sin \delta$$

Consequently, for the sinusoidally varying case,

$$P_0 = \omega (B_m H_m / 2) \sin \delta = \omega \mu_0 H_{eff}^2 \mu'' \tag{3.124}$$

Here, H_{eff} is the actual (effective) field strength, and B and H are the instantaneous parameters. In deriving (3.124), we have taken into consideration that

$$B_m = \mu_0 |\underline{\mu}| H_m \text{ and } |\underline{\mu}| \sin \delta = \mu'' \tag{3.125}$$

In reality, the elliptic loop occurs rarely, mainly at low amplitudes of the field or at high frequencies. Fig. 3.20 illustrates how the hysteresis loop approaches the elliptic form as the frequency of weak fields grows.

Equivalent parameters. The complex permeability and elliptic loop enable us to evaluate approximately the properties of ferromagnets through the parameters

which are equivalent to a certain degree. The equivalent parameters are most often chosen so that at the specified values of ω and B_m the actual and equivalent ferromagnets can dissipate the identical amounts of power at the same values of H_m.

The equivalent parameters are easy to determine. Thus, knowing P_0, B_m, and H_m, we find from (3.125)

$$|\underline{\mu}| = |\mu' - j\mu''| = B_m/\mu_0 H_m \tag{3.126}$$

and from (3.124)

$$\mu'' = P_0/\omega\mu_0 H_{eff}^2 \tag{3.127}$$

and, finally,

$$\mu' = \sqrt{|\underline{\mu}|^2 - (\mu'')^2} \tag{3.128}$$

Proceeding from the parametric equations of an ellipse, we can determine its axes $2a$ and $2b$ (see Fig. 3.19) and plot the entire ellipse in coordinates B and H.

Resonant effects. The carriers of a magnetic moment in ferromagnetic materials are liable to free damped oscillations at natural frequencies. Obviously, this effect can give rise to resonances.

In 1913, V.K. Arkadyev investigated for the first time the dependence of permeability on frequency over the SHF range. He suggested that this dependence stemmed from weakly defined resonant effects. The effects were indeed weak, and it was very difficult to observe resonance phenomena at the time in the absence of high-sensitivity devices; what is more, the materials under investigation were iron and nickel. Modern devices enable clear observation of these effects, which find substantial uses in super-high-frequency applications.

Resonance occurs in ferrites, garnets of yttrium, and similar materials of negligible conductivity. The simplest way to produce and observe resonance is to subject a ferromagnetic material to the action of a high external dc field along with a weak magnetic field at high frequencies; intrinsic fields in anisotropic media can produce the same effect.

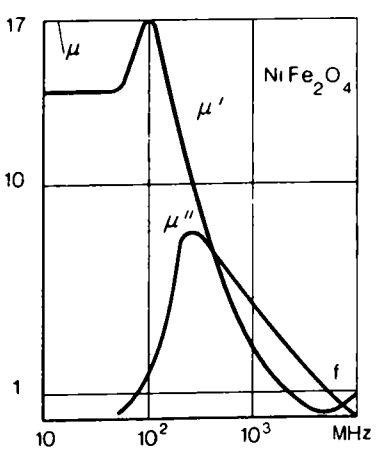


Fig. 3.21

Resonant effects in ferrites can also appear in the absence of an external field (natural resonance) at frequencies of a few million hertz, though the quality factor of the oscillatory system in this case is not high. But the existence of resonance can be judged of with confidence from a small increase in the real part of permeability μ' with a rise in frequency (Fig. 3.21). Such a phenomenon is unlikely to take place if the process described by the first-order equation is of the relaxation type. Resonant effects may only be evident if the magnetic state of a substance corresponds to a differential equation of at least the second order.

The time lag of the process that results in the second derivative in magnetization equations may be accounted for by the relation between magnetization and elastic mechanical deformation of the body being magnetized.

3.5. Notes on Sections 3.1 to 3.4

From the text of Ch. 1, which defines field vectors in vacuum, and from the detailed analysis of the present chapter it follows that vectors **E** and **B** are just *the vectors of an electromagnetic field* and vectors **P** and **M** are *the vectors characterizing the state of a polarized substance*, i.e. its polarization. The vectors

$$\mathbf{D} = \varepsilon_0\mathbf{E} + \mathbf{P} \text{ and } \mathbf{H} = (1/\mu_0)\mathbf{B} - \mathbf{M}$$

represent a linear combination of the respective pairs of field and polarization vectors; the formulation provides for convenience of describing the field in polarized media. The equations repeated in this section serve as the definitions of vectors **D** and **H**. The analogy stressed out here is in agreement with that presented in modern relativistic electrodynamics.

The above reasoning needs one stipulation: in considering the field in a polarized medium from the viewpoint of macroscopic theory, the vectors **B** and **E** are the averaged values of field vectors as are the vectors **P**, **M**, **D**, and **H**.

3.6. Electric Field in a Conducting Medium. Quasistatic Conditions

As noted earlier, conductors differ from dielectrics in that the former have free charges capable of moving under the action of an electric field. In conductors, therefore, an electric field causes an ordered flow of electric charges, which is an electric current. In conductors of finite and constant conductivity the process of charge motion is defined by the current density proportional to field strength (Ohm's law):

$$\mathbf{J} = \sigma\mathbf{E}$$

where σ is conductivity.

At the interface of two immobile conductors

$$\sigma_1 E_{1n} = \sigma_2 E_{2n} \tag{3.129}$$

As is clear from the formula, the normal components of current density are equal since the accumulation of charges does not occur ($\partial E/\partial t = 0$). The condition (3.129) is a particular case of the more general relation (3.62). Statements (3.28) and (3.63) given earlier in the text allow us to affirm that

$$E_{1t} = E_{2t} \tag{3.130}$$

The similarity between boundary conditions for conductors and dielectrics, and even for magnetic materials, is put to extensive use in investigating fields on models.

Along with ordinary conductors obeying Ohm's law, of substantial importance in modern electrical engineering are (1) plasmas which are highly ionized gases, (2) superconductors ($1/\sigma = 0$) and (3) nonlinear semiconductors showing distinctive features in the region of boundaries between two materials, and also anisotropic crystalline substances which do not obey simple equations of electrodynamics. Like the physics of dielectric and ferromagnetic materials, the physics of the above substances and materials lies beyond the established bounds of the course in the theoretical bases of electrical ergineering.

Chapter 4 SIMPLE SOLUTIONS OF FIELD EQUATIONS AND THEIR APPLICATIONS IN CIRCUIT THEORY

It is possible to find a number of solutions of the field equations, applicable on circuit theory, without resorting to the complete calculation of a field. Some of the solutions are set forth in this chapter.

In particular, the chapter shows the way of defining in simple cases the integral circuit parameters such as C, M, L, and R. The subsequent sections describe the principle of reciprocity for circuits. The text also presents the techniques of calculation of flux linkages in electric circuits due to magnetized bodies and, besides, explains the calculation of the current in a circuit due to changes in the polarization of a body immersed in a capacitor field or the current due to motion of an electric charge between capacitor plates.

4.1. Electric Field and Electric Parameters of Parallel Wires

The field of a uniformly charged thin straight wire (electric, or charged axis) is very easy to define by applying Gauss' theorem:

$$E = E_r = -\partial\varphi/\partial r = \tau'/r \tag{4.1}$$

and

$$\varphi = -\tau' \int dr/r = -\tau' \ln r + \text{constant} \tag{4.2}$$

where $\tau' = \tau/2\pi\varepsilon_0$; r is the radius in the cylindrical coordinate system.

For two wires (Fig. 4.1) carrying charges $+\tau$ and $-\tau$ per unit length, the potential at a point which is a distance r_+ and r_- remote from the respective wires is given by

$$\varphi = \tau' \ln (r_-/r_+) \tag{4.3}$$

Here the constant is so chosen that all the charges, or points $\pm\tau$, equidistant from the axes, have the zero potential.

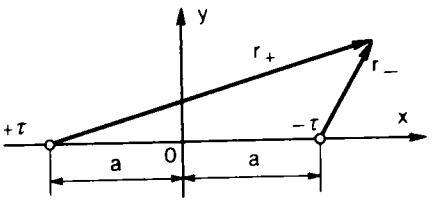


Fig. 4.1

Expressing r_- and r_+ in cartesian coordinates, we obtain

$$\varphi = \tau' \ln k \tag{4.4}$$

where

$$k = r_-/r_+ = \sqrt{(x-a)^2 + y^2}/\sqrt{(x+a)^2 + y^2}$$

The equation for equipotentials, φ = constant, then results from the condition k^2 = constant if

$$(x-a)^2 + y^2 = k^2(x+a)^2 + k^2y^2$$

The above expression is the equation of circles

$$(x - x_0)^2 + y^2 = R^2$$

whose centers lie on the x axis (i.e. at $y = 0$) and are a distance x_0 remote from the origin of coordinates. Here,

$$x_0^2 = a^2[(k^2+1)/(k^2-1)]^2 = a^2[1 + 2/(k^2-1)]^2 \tag{4.5}$$

The radii of these circles are

$$R^2 = x_0^2 - a^2 = 4k^2a^2/(k^2-1)^2 \tag{4.6}$$

The potential difference between two equipotentials, φ_1 and φ_2, with radii R_1 and R_2 (Figs. 4.2 and 4.3) is

$$U_{12} = \varphi_1 - \varphi_2 = \tau' (\ln k_1 - \ln k_2) \tag{4.7}$$

where $k_{1,2} = r_{1,2-}/r_{1,2+}$.

We have described the equipotentials encircling the fields, the position of charged axes being given. Imagine now that the equipotential surface, i.e. the circle φ_2 of Fig. 4.2, is the cross section of a cylindrical wire of radius R_2 lying inside a hollow cylinder of internal radius $R_1 > R_2$. The latter has a potential φ_1. How can we determine the field between the cylinders and also the charge τ?

Electrical engineering practice teaches that in a homogeneous linear medium the same field always sets in between the electrodes of the same geometrical shape and with the same potential difference. What follows from the above statement is that one should always expect to have the unique solution to the field equations in question. In other words, any solution that satisfies field equations and boundary conditions of the problem is complete and unique. We can also prove the uniqueness of the solution mathematically (see Sec. 5.1).

Applying the same reasoning to the region bounded by two equipotential cylinders (see Fig. 4.2) and knowing that (4.3) is precisely the field equation, we are in a position to assume that the field of given wires is identical to the field of charged axes,

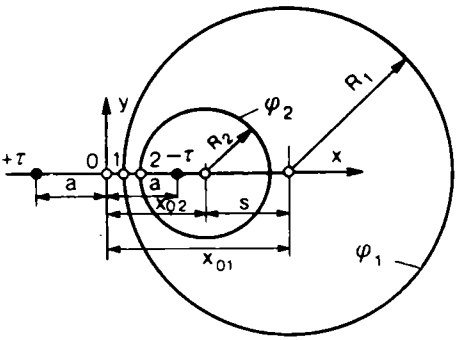


Fig. 4.2

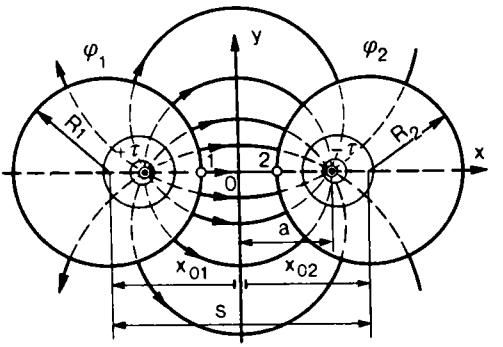


Fig. 4.3

$\pm\tau$, if the latter field occupies the requisite position. Using (4.5) and (4.6) we can express three parameters x_{01}, x_{02}, and $2a$, being additionally required here, in terms of the specified parameters R_1, R_2, and s. If $s < R_1 - R_2$ (see Fig. 4.2), the distance between the geometrical axes of cylinders is

$$s = x_{01} - x_{02} \tag{4.8}$$

Rewriting (4.6) for R_1 and R_2 gives

$$R_1^2 = x_{01}^2 - a^2 \quad \text{and} \quad R_2^2 = x_{02}^2 - a^2 \tag{4.9}$$

Solving the last three equations simultaneously yeilds

$$x_{02} = (R_1^2 - R_2^2 - s^2)/2s \tag{4.10a}$$

$$x_{01} = (R_1^2 - R_2^2 + s^2)/2s \tag{4.10b}$$

$$a = \sqrt{x_{02}^2 - R_2^2} = \sqrt{x_{01}^2 - R_1^2} \tag{4.11}$$

The parameters $k_1 = r_{1-}/r_{1+}$ and $k_2 = r_{2-}/r_{2+}$ that determine wire potentials are easy to find from Fig. 4.2 as the ratio between the distances from the axes $-\tau$ and $+\tau$ to cylinder points *1* and *2* nearest to the origin 0:

$$\begin{aligned} k_1 &= (a - x_{01} + R_1)/(a + x_{01} - R_1) \\ k_2 &= (a - x_{02} + R_2)/(a + x_{02} - R_2) \end{aligned} \tag{4.12}$$

After the position of electric axes has been found, we can readily determine the potential difference at a given τ and calculate the capacitance. The potential difference between the cylinders (wires) is given by

$$\begin{aligned} U_{12} &= \tau' (\ln k_1 - \ln k_2) \\ &= \tau' \ln \frac{(a - x_{01} + R_1)/(a + x_{01} - R_1)}{(a - x_{02} + R_2)/(a + x_{02} - R_2)} \end{aligned} \tag{4.13}$$

By definition, the capacitance per unit length is

$$C_0 = \tau / U_{12} = 2\pi\varepsilon_0 / \ln(k_1/k_2) \tag{4.14}$$

A similar discourse can apply for two wires of radii R_1 and R_2 (see Fig. 4.3). Instead of (4.8), here we have

$$s = x_{01} + x_{02} \tag{4.15}$$

where x_{01} and x_{02} are the magnitudes of distances.

In contrast to (4.10), now we obtain

$$x_{01} = (s^2 + R_1^2 - R_2^2)/2s \tag{4.16a}$$

$$x_{02} = (s^2 + R_2^2 - R_1^2)/2s \tag{4.16b}$$

The quantity a is expressed by (4.11) as before.

The parameters k_1 and k_2 defining the cylinder potentials are easy to find for points *1* and *2* lying on the x axis of coordinates:

$$k_1 = (a + x_{01} - R_1)/(a + R_1 - x_{01}) \tag{4.17a}$$

$$k_2 = (a + R_2 - x_{02})/(a + x_{02} - R_2) \tag{4.17b}$$

The potential difference is expressed by the companion formula to (4.13)

$$U_{12} = \tau' \ln \frac{k_1}{k_2} = \tau' \ln \frac{(a + x_{01} - R_1)(a + x_{02} - R_2)}{(a - x_{01} + R_1)(a - x_{02} + R_2)} \tag{4.18}$$

In a particular case, for wires of the same radius $R = R_1 = R_2$,

$$U_{12} = 2\tau' \ln[(a + x_0 - R)/(a - x_0 + R)] \tag{4.19}$$

The capacitance here is

$$C_0 = \tau/U_{12} = \pi\varepsilon_0/\ln[(a + x_0 - R)/(a - x_0 + R)] \approx \pi\varepsilon_0/\ln(s/R) \tag{4.20}$$

Wire above a conducting plane. If a wire runs parallel to a conducting plane (in particular, above ground), we can consider its field proceeding from the assumption that the conducting plane is the surface of a cylinder whose radius R approaches infinity. If h is the distance from the geometrical axis of the wire of radius $R = R_1$ to the plane (Fig. 4.4*a*), then the field can be specified as the field of two charged axes with coordinates

$$a = \sqrt{h^2 - R^2} \tag{4.21}$$

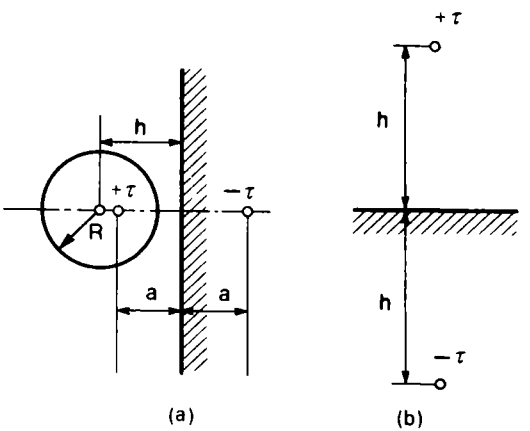


Fig. 4.4

where a is the distance counted off in the direction normal to the plane. In this case the whole charge $+\tau$ lies on the axis running inside the wire, and the charge $-\tau$ on the other axis.

Formula (4.21) follows from (4.11) since the mirror image of the wire has the same radius R and the distance between the image axis and the wire axis is $s = 2x_0 = 2h$.

At $h \gg R$, the electric axis practically coincides with the geometric axis, i.e. $a \approx h$, as is the case with the wire suspended over the ground plane (Fig. 4.4*b*), for example, a wire of $R = 1$ cm at a height of $h = 15$ m.

Potential and charge distribution in a system of conducting bodies. So far we have discussed the field of one or two charged bodies. But engineering had often to deal with systems consisting of a number of wires (conducting bodies) which accept different potentials and carry different charges.

Coefficients of potential. Using the principle of superposition, we can find the potential of the system of charged wires at any point A as the sum of potentials due to charges on the first, second, third wire, etc.:

$$\varphi_A = \varphi_{A1} + \varphi_{A2} + \varphi_{A3} + \ldots$$

where each component is in direct proportion to the respective charge, i.e.

$$\varphi_{A1} = \alpha_{A1} q_1, \varphi_{A2} = \alpha_{A2} q_2 \ldots$$

The coefficients of potential, $\alpha_{A1}, \alpha_{A2} \ldots$, depend both on the location of point A and on the geometry of all wires carrying charges. Combining the last formulas, we can obtain the potential at A:

$$\varphi_A = \alpha_{A1} q_1 + \alpha_{A2} q_2 + \alpha_{A3} q_3 + \ldots$$

Assuming that the point A is first located on wire *1*, then on wire *2*, etc., we set up the system of equations:

$$\begin{aligned} \varphi_1 &= \alpha_{11} q_1 + \alpha_{12} q_2 + \ldots \\ \varphi_2 &= \alpha_{21} q_1 + \alpha_{22} q_2 + \ldots \\ &\ldots\ldots\ldots\ldots\ldots \end{aligned} \tag{4.22}$$

or

$$\varphi_k = \alpha_{ki} q_i \tag{4.22a}$$

where φ_1 is the potential on the first wire; φ_2 is the potential on the second wire, etc., and the coefficients α_{ki} are the coefficients of potential. Formula (4.22a) represents the system of equations written in matrix notation which implies summation with respect to the identical indexes i.

The matrix equation enables us to solve directly the problem in the distribution of potentials in the systems of wires if we know the charges they carry and coefficients α_{ki}.

Coefficients of capacitance. One often needs to solve an inverse problem: to find the charges on wires, given the potentials. For this it is necessary to solve matrix equation (4.22) with respect to charges:

$$q_i = \beta_{ik} \varphi_k \tag{4.23}$$

Such a system is known as the system of equations in matrix notation with the coefficients of capacitance, β_{ik}. The relation between the coefficients of capacitance and potential is easy to find regarding (4.23) as the solution of (4.22), in which case

$$\beta_{ki} = 1/\alpha_{ik} = A_{ki}/D \tag{4.23a}$$

where the coefficients A_{ki} are cofactors, or adjuncts, and D is the determinant of matrix (4.22).

Direct (partial) capacitances. Essentially the same relationships can be written in another, very convenient notation using the concept of a direct capacitance between two conductors. Direct capacitances thus interrelate pairwise all the conductors of the system under study. As an example, Fig. 4.5 illustrates direct capacitances in a triple-conductor cable.

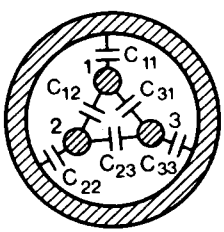


Fig. 4.5

Given the connection diagram of direct capacitances, we can easily express the charge on each of the wires in terms of the potentials and capacitances. Thus assuming the potential of the cable sheath equal to zero, we derive the expression for charges on the three conductors of the cable:

$$\begin{aligned} q_1 &= \varphi_1 C_{11} + (\varphi_1 - \varphi_2)C_{12} + (\varphi_1 - \varphi_3)C_{13} \\ q_2 &= (\varphi_2 - \varphi_1)C_{21} + \varphi_2 C_{22} + (\varphi_2 - \varphi_3)C_{23} \\ q_3 &= (\varphi_3 - \varphi_1)C_{31} + (\varphi_3 - \varphi_2)C_{32} + \varphi_3 C_{33} \end{aligned} \tag{4.24}$$

Capacitances C_{12} and C_{21}, C_{23} and C_{32} are identically equal. Equating the coefficients at identical potentials in matrix equations of the types such as (4.23) and (4.24), we readily obtain:

$$\beta_{11} = C_{11} + C_{12} + C_{13}, \beta_{23} = -C_{23}, \text{etc.}$$

or, in the general case, for the $(n + 1)$th conductor:

$$\begin{aligned} \beta_{ii} &= C_{i1} + C_{i2} + \ldots + C_{in} \\ \beta_{ik} &= -C_{ik} \quad \text{at} \quad i \neq k \end{aligned} \tag{4.25}$$

The solution of these equalities with respect to C_{ii} gives

$$C_{ii} = \beta_{i1} + \beta_{i2} + \ldots + \beta_{ii} + \ldots + \beta_{in} \tag{4.26}$$

Reciprocity principle for the coefficients of potential and capacitance. For the system of a potential field of charged bodies the principle of reciprocity holds true. This principle is expressed by the equalities

$$\alpha_{ik} = \alpha_{ki} \text{ and } \beta_{ik} = \beta_{ki} \tag{4.27}$$

In words it reads: the potential of one conductor due only to a charge on the other is equal to the potential on the second due only to a charge on the first.

The equality of capacitance coefficients $\beta_{ik} = \beta_{ki}$ is in agreement with the obvious equality of direct capacitances $C_{ik} = C_{ki}$; in point of fact, C_{ik} and C_{ki} differ only in notation and indeed represent the same capacitance between wires k and i. The equality of potential coefficients, $\alpha_{ik} = \alpha_{ki}$, directly follows from the equality of capacitance coefficients since α_{ik} can be found from the solution of system (4.23) for which the algebraic adjuncts are $B_{ki} \equiv B_{ik}$ (because $\beta_{ik} = \beta_{ki}$).

The coefficients of potential for overhead lines. For the system of parallel aerial conductors (Fig. 4.6) suspended above the ground, the coefficients α_{ik} are easy to calculate: Really, according to formula (4.3), the potential on the surface of one conductor due to a charge on this conductor, τ_1, and its mirror image is given by

$$\varphi_{11} = \tau_1' \ln(2h_1/r_1) \tag{4.28}$$

where h_1 is the height of suspension of conductor *1* above the ground; r_1 is the radius of this conductor; and $\tau_1' = \tau_1/2\pi\varepsilon_0$ is the charge per unit length of conductor.

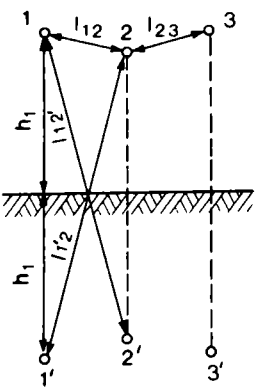


Fig. 4.6

The potential of the same first conductor due to a charge on the second conductor and its mirror image is

$$\varphi_{12} = \tau_2' \ln(l_{12'}/l_{12}) \tag{4.29}$$

where $\tau_2' = \tau_2/2\pi\varepsilon_0$ and τ_2 is the charge per unit length of the second conductor; $l_{12'}$ is the distance between conductor *1* and the mirror image of conductor *2*; l_{12} is the distance between conductors *1* and *2* (see Fig. 4.6).

Similarly, we find

$$\varphi_{13} = \tau_3' \ln(l_{13'}/l_{13}) \tag{4.30}$$

where the designators are analogous to the above ones.

Thus the potential coefficients for the system of Fig. 4.6 with the normalized charges $\tau_k' = \tau_k/2\pi\varepsilon_0$ are

$$\alpha_{11} = \ln(2h_1/r_1)$$

$$\alpha_{12} = \ln(l_{12'}/l_{12})$$
$$\alpha_{13} = \ln(l_{13'}/l_{13}) \tag{4.31}$$

The coefficients in the expression for the potential on conductor *2* are determined in exactly the same manner.

Formulas with coefficients α_{ik},β_{ik} and C_{ik} closely resemble the general expressions for distribution of currents and potentials in linear circuits. In the case of periodically varying potentials, the resemblance becomes still greater on passing to complex quantities and multiplying by $j\omega$ because $j\omega\tau = \dot{I}_k$. This similarity is useful to keep in mind; proceeding from this, we can apply all the techniques of calculation of linear circuits to the system of equations being considered, in particular, when determining the coefficients α_{ik}, β_{ik}, and direct capacitances C_{ik}.

The systems of equations with the coefficients of potential and capacitance are often referred to as Maxwell's formulas because they were systematized by Maxwell in his treatise. Notice that the systems of equations for an electric circuit with loop currents and nodal potentials were also given by Maxwell.

High-voltage overhead line wires. We can readily determine the upper limit of voltage between wires for a double line, given the wire radius, in particular, if we can neglect the effect of ground. Note that we mean here only the calculations made with respect to a maximum permissible field strength. Actually, the technical calculations are more complex.

Let r_0 be the wire radius and s the distance between the wire axes. Assuming $s = 10$ m and $2r_0 = 3.3$ cm, we may disregard the displacement of axes ($x_0 \approx a$ and $s \approx 2a$). The voltage between wires is then

$$U_{12} = (\tau/\pi\varepsilon_0)\ln(s/r_0)$$

The maximum field strength on the wire surface is

$$E_{\max} \approx \tau/2\pi\varepsilon_0 r_0$$

if we neglect a nonuniform charge distribution over the wire surface due to the influence of neighboring wires, line supports, and ground. The field strength must be less than 30 kV/cm to avoid an electric discharge in the air, i.e. a corona as a glowing layer round the wires. Of course, the emergence of a corona does not yet mean that insulation between the wires breaks down, but this discharge may be a cause of its subsequent rupture. Also, a corona discharge entails power loss, and gives rise to upper harmonics of the current, which are highly undesirable because they interfere with the operation of communication systems. That is why the actual value of a sinusoidally varying field must be lower than the limiting value of $E_{lim} = 20$ kV/cm.

In this case, $\tau/\pi\varepsilon_0 = 2r_0E_{lim}$ and

$$U_{lim} = 2r_0E_{lim}\ln(s/r_0) \tag{4.32}$$

Thus at $s = 10$ m and $2r_0 = 3.3$ cm, the permissible effective voltage between the wires is $U_{lim} = 420$ kV. With an increase in the distance s between the wires of the same diameter to 15 m, U_{lim} rises to merely 450 kV. In use are now overhead lines transmitting high power over long distances at nominal voltage ratings of 500 to

750 kV. The trend today is toward overhead lines for still higher voltage ratings. It is then hardly reasonable to employ single conductors of increased diameter. As regards overhead lines rated at very high voltages, the approach is to replace a separate single-phase conductor by a group of three or four parallel wires or stranded conductors spaced a certain distance apart but electrically connected one to another (Fig. 4.7). Mutual shielding of conductors lowers a maximum field strength on the surface of individual wires.

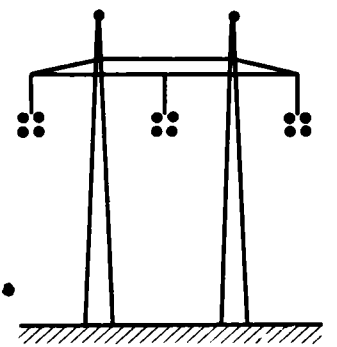


Fig. 4.7

Such a separation of a conductor into a number of wires in high-voltage installations was suggested by V.F. Mitkevich in 1910, and the results of his investigations and experiments were published the same year.

4.2. Self-Inductance and Mutual Inductance

Basic definitions. If two loops of any electric circuit lie in sufficiently close proximity to each other, the magnetic field due to the current in the first loop can spread over the region of location of the second loop, as is the case with two parallel double lines (Fig. 4.8) or two coaxial coils (Fig. 4.9). As a result, when the current flows only in the first loop ($i_1 \neq 0, i_2 \neq 0$), the second one proves linked by the magnetic flux pro-

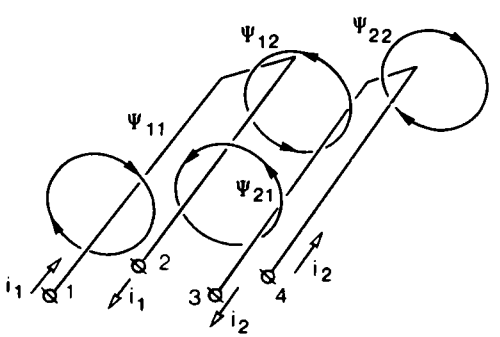


Fig. 4.8

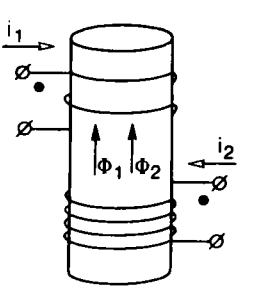


Fig. 4.9

portional to the first current:

$$\Psi_{21} = M_{21} i_1 \tag{4.33}$$

The flux that links the first loop, Ψ_{11}, is equal here to the product of the first current and self-inductance L_1:

$$\Psi_{11} = L_1 i_1 \tag{4.34}$$

Similarly, when current exists only in the second loop, the flux linking the first loop is

$$\Psi_{12} = M_{12} i_2 \tag{4.35}$$

and the internal flux linkage of the second loop is

$$\Psi_{22} = L_2 i_2 \tag{4.36}$$

Figure 4.8 illustrates schematically all flux components.

This section opens with the sentence where we have used the phrase "close proximity". How must we understand it here? As the space between the loops increases, the magnetic linkage between them decays so that $M_{21} i_1$ and $M_{12} i_2$ can be ignored in the analysis, otherwise we must say that the loops lie rather close to each other.

If the current in the first loop changes, an electromotive force is induced in the second:

$$\mathcal{E}_2 = -M_{21} di_1/dt \tag{4.37}$$

In a similar manner, an emf is induced in the first loop with a change of current only in the second loop:

$$\mathcal{E}_1 = -M_{12} di_2/dt \tag{4.38}$$

This phenomenon is known as mutual induction, and the proportionality coefficients

$$M_{12} = M_{21} = M \tag{4.39}$$

are termed the coefficients of mutual induction, or just mutual inductances. The principle of reciprocity (4.39), commonly known from physics, rigorously follows from the energy equation for inductively coupled loops [see Eqs. (4.51) and (4.52)]. The unit of mutual inductance and self-inductance is the henry (H).

If the current flows both in the first and in the second loop, the total flux linkage is the algebraic sum of flux linkages due to self-inductance and mutual inductance (see Fig. 4.8):

$$\begin{aligned} \Psi_1 &= \Psi_{11} + \Psi_{12} = L_1 i_1 + M_{12} i_2 \\ \Psi_2 &= \Psi_{22} + \Psi_{21} = L_2 i_2 + M_{21} i_1 \end{aligned} \tag{4.40}$$

Here Ψ_1 and Ψ_2 are the total flux linkage of the first and the second loop respectively. The formulas are set up on the assumption that the flux linkage of one loop with a positive current rises under the influence of the positive current in the other (see Figs. 4.8 and 4.9). The above statement places a definite requirement on the consistent choice of positive current directions.

Marking the ends of inductively coupled loops. To determine the sense of the flux Mi_2 or Mi_1 as it adds to the flux $L_1 i_1$ or $L_2 i_2$, it is necessary to mark the terminals of branches with inductively coupled loops, assuming L_1, L_2, and M positive.

A terminal (pole) of one of the two inductively coupled elements (or loops) is usually taken as the start and marked with a certain symbol (more often with a dot)

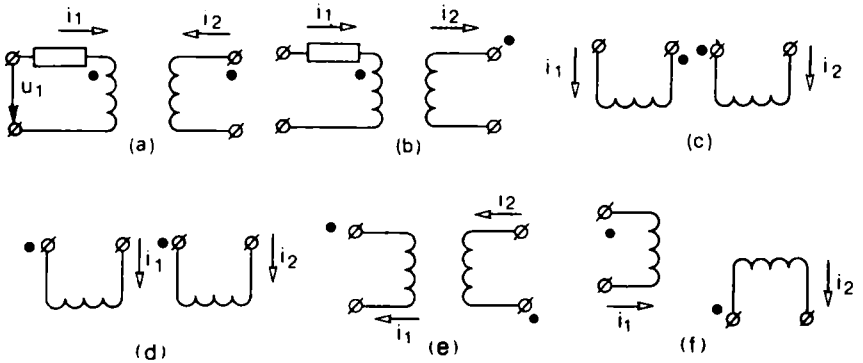


Fig. 4.10

put near the terminal (see Figs. 4.9 and 4.10). The second mark is drawn on the second element to indicate that the flux linkage of each element rises as a result of mutual induction when the currents i_1 and i_2 flow in the same sense with respect to the marked poles of the elements (see Fig. 4.9 and 4.10*a*,*c*,*d*). In this case

$$\Psi_1 = L_1 i_1 + M i_2, \Psi_2 = L_2 i_2 + M i_1 \tag{4.41}$$

If the currents in two elements are in the opposite sense with respect to the marked poles (Fig. 4.10*b*,*e*,*f*), then the flux linkage of each element is expressed differently:

$$\Psi_1 = L_1 i_1 - M i_2, \Psi_2 = L_2 i_2 - M i_1 \tag{4.42}$$

As regards the two lines of Fig. 4.8 or coaxial coils of Fig. 4.9 with the known direction of winding, the manner in which we must put polarity markings is clear. Where the relative orientation of magnetic fluxes is not known definitely, the measurement of induced voltage can give an idea of how to mark the poles: if the currents are in the same direction with respect to polarity markings, the self-induction voltage

$$u_{L_1} = L_1 di_1/dt \text{ or } \dot{U}_{L_1} = L_1 j\omega \dot{I}_1 \tag{4.43}$$

must add to the voltage of mutual induction, and therefore

$$\begin{aligned} u_{(L,M)_1} &= L_1 di_1/dt + M di_2/dt \\ u_{(L,M)_2} &= M di_1/dt + L_2 di_2/dt \end{aligned} \tag{4.44}$$

Correspondingly, for the circuit with an ac current we can write

$$\begin{aligned} \dot{U}_{(L,M)_1} &= L_1 j\omega \dot{I}_1 + M j\omega \dot{I}_2 \\ \dot{U}_{(L,M)_2} &= M j\omega \dot{I}_1 + L_2 j\omega \dot{I}_2 \end{aligned} \tag{4.45}$$

On the contrary, if the currents i_1 and i_2 are in the opposite direction with respect to the marked poles, then

$$\begin{aligned} u_{(L,M)_1} &= L_1 di_1/dt - M di_2/dt \\ u_{(L,M)_2} &= -M di_1/dt + L_2 di_2/dt \end{aligned} \tag{4.46}$$

In Fig. 4.10*a* and *b* the paths of inductively coupled loops have a resistance.

Therefore, for the circuit of Fig. 4.10*a* carrying the first current,

$$u_1 = i_1R_1 + L_1di_1/dt + Mdi_2/dt \tag{4.47}$$

and for the circuit of Fig. 4.10*b*,

$$u_1 = i_1R_1 + L_1di_1/dt - Mdi_2/dt \tag{4.48}$$

Energy stored in a magnetic field of two inductively coupled loops. Find the energy stored in the field of two circuits from the source energy consumed on condition that initially $i_1 = i_2 = 0$.

Let us connect only the first circuit to the source leaving the second open ($i_2 = 0$). As the current grows from zero to i_1 during time t_1, the first source must supply the energy

$$W_1 = \int_0^{t_1} i_1^2R_1dt + \int_0^{t_1} i_1L_1(di_1dt)dt \tag{4.49}$$

Substracting the energy dissipated in the resistance R_1 (the first term) from W_1, we find that the first source has given up the following amount of energy to the magnetic field:

$$w_1 = L_1i_1^2/2$$

Connect the second circuit to a new source at an instant t_1 to allow the current in the circuit to rise from zero to i_2 by the instant t_2. If the current in the first circuit remains constant (i_1), the energy supplied to the second circuit by its source and stored in the magnetic field must be

$$w_2 = \int_{t_1}^{t_2} i_2L_2(di_2/dt)dt = L_2i_2^2/2$$

But an emf of mutual induction is induced in the first circuit as the current i_2 grows; in order for the first current to remain constant, the first source must maintain the voltage

$$u_1 = i_1R_1 + M_{12}di_2/dt$$

that balances out not only the voltage drop across the resistance (i_1R_1) but also the emf induced in the first circuit. Therefore, the energy the first source gives up to the magnetic field during the growth of the second current is

$$w_{12} = \int_{t_1}^{t_2} i_1u_1di - \int_{t_1}^{t_2} i_1^2R_1dt = i_1\int_{t_1}^{t_2} M_{12}(di_2/dt)dt = M_{12}i_1i_2 \tag{4.50}$$

The total energy stored in the magnetic field at currents i_1 and i_2 in two inductively coupled circuits is then

$$w = L_1i_1^2/2 + L_2i_2^2/2 + M_{12}i_1i_2 \tag{4.51}$$

If we switched on the second source and then the first, we would find from the analysis that

$$w = L_1i_1^2/2 + L_2i_2^2/2 + M_{21}i_1i_2 \tag{4.52}$$

The final conditions of the circuits are the same in both cases: a magnetic field exists in the system, which lends itself to a unique definition for the given configuration

of wires at the given currents. This statement leads us to conclude that mutual inductances are equal:

$$M_{12} = M_{21} = M \tag{4.53}$$

Circuit containing inductively coupled elements. The above discourse on two inductively coupled loops applies well to a system consisting of three or more loops. The flux linkage of each of the loops can be given by the equalities

$$\begin{aligned} \Psi_1 &= M_{11}i_1 + M_{12}i_2 + M_{13}i_3 + \dots \\ \Psi_2 &= M_{21}i_1 + M_{22}i_2 + M_{23}i_3 + \dots \\ \Psi_3 &= M_{31}i_1 + M_{32}i_2 + M_{33}i_3 + \dots \end{aligned} \tag{4.54}$$

if the currents $i_1, i_2, \dots$ are in the same direction with respect to the marked poles of inductive elements. If the sense of any current is different, the respective term changes sign, for example, $-M_{mn}i_n$ instead of $+M_{mn}i_n$.

In the above system of equalities, all the coefficients with two identical indexes are inductances of the respective loops, i.e. $M_{11} = L_1$, $M_{22} = L_2$, etc., and the coefficients with two different indexes are mutual inductances of the respective pairs of loops. Applying the same reasoning as given above to any pair of loops, it can be shown that the following equality is always valid:

$$M_{mn} = M_{nm} \tag{4.55}$$

Positive and negative mutual inductances. In the general case, mutual inductances can be both positive and negative. According to the rules of algebra, we can reveal this only by substituting numerical values. Thus, in (4.54) the mutual inductance M_{12} may be negative and equal to, say, -0.14 mH.

For two loops defined by one mutual inductance, pole marking is made so that M is positive in value, though here, too, a controllable mutual inductance usually takes both positive and negative values. Thus in two ring-type windings, one of which can turn by 180°, M changes from $+|M_0|$ to $-|M_0|$. Pole marking is certainly necessary in this case too.

For a number of loops, say, three loops with L_1, L_2, and L_3 linked together, it is expedient to suppose that M_{12}, M_{23}, and M_{31} can be both positive and negative; the sign of mutual inductance can certainly be determined only after pole marking. It should be noted that in a number of cases it is impossible to mark poles so that all mutual inductances M_{mn} can be positive.

Figure 4.11 illustrates schematically three large coils disposed one next to the other; the arrangement of coils is approximately the same as that for reactors used in power systems to limit short-circuit currents. The directions of fluxes Φ_1, Φ_2, and Φ_3 are shown to be positive for each of the coils. This is exactly the direction pattern for the flux due to the positive current in each reactor. The figure also shows the flux due to the current in the first reactor, which links the turns of the second and the third reactor. When this flux is due to the positive current of the first reactor, the fluxes Φ_{21} and Φ_2 are in opposite direction as are Φ_{31} and Φ_3.

Assuming the upper terminals (poles) to be the start of all windings and marking them with dots as shown in Fig. 4.11, we find that, with the positive currents heading for the marked terminals, the flux linkage of reactors is expressed as

$$\Psi_1 = i_1L_1 + i_2M_{12} + i_3M_{13}$$

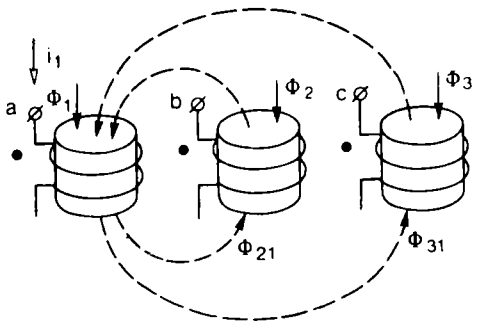


Fig. 4.11

$$\Psi_2 = i_1 M_{21} + i_2 L_2 + i_3 M_{23} \qquad (4.56)$$
$$\Psi_3 = i_1 M_{31}\, i_2 M_{32} + i_3 L_3$$

where

$$M_{12} = M_{21}, M_{23} = M_{32}, M_{31} = M_{13},$$

In this case all mutual inductances are negative, and there is no way of marking the poles of these three coils with one sign (dot) in order that the three inductances might have positive values.

Both self-inductance L and mutual inductance M are important parameters of an electric curcuit. The calculation of inductance comes to determining a magnetic flux linking the given loop; in other words, it calls for the calculation of a magnetic field. In a number of cases it is convenient to evaluate the magnetic flux not directly, as the vector flux of magnetic induction, but as the closed-loop integral of vector potential

$$\Phi = \oint \mathbf{A} d\mathbf{l}$$

Parallel wires. Using Eq. (2.87) derived earlier, we determine the vector potential in the field of two wires carrying currents $+i$ and $-i$:

$$A = A_z = (\mu_0 i/2\pi)\ln(r_-/r_+) \qquad (4.57)$$

where the z axis (Fig. 4.12) is the axis assumed to run in the positive direction; r_- and r_+ are the distances from respective wires to the observation point. This expression bears much resemblance to (4.3) for the electric potential φ of two charged axes.

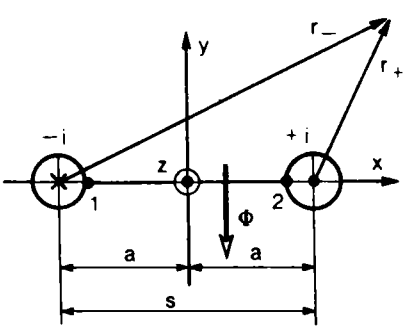


Fig. 4.12

According to (2.77), the magnetic flux linking any loop is

$$\Phi = \oint \mathbf{A} d\mathbf{l} \tag{4.58}$$

So, using (4.57), we can readily find the flux going through the air between the two wires of the same radius r_0:

$$\begin{aligned}\Phi &= A_1 - A_2 = \mu_0 i/2\pi \ln [(r_-/r_+)_1 (r_+/r_-)_2] \\ &= (\mu_0 i/2\pi) \ln [(s - r_0)/r_0]^2 = (\mu_0 i/\pi) \ln [(s - r_0)/r_0]\end{aligned}$$

Here $s = 2a$ is the distance between the wire axes as shown in Fig. 4.12. This is the intercept of the x axis that lies between points *1* and *2* denoting the wires carrying the currents into the paper and out of the paper $-i$ and $+i$ respectively.

The air inductance for a two-wire line per unit length is

$$L_0 = \Phi/i = (\mu_0/\pi) \ln [(s - r_0)/r_0] \approx (\mu_0/\pi) \ln (s/r_0) \tag{4.59}$$

This formula expresses only the external (air) inductance. The effect of the magnetic field inside a conductor is discussed in Secs. 6.3 and 7.3.

It is worth comparing (4.59) with (4.20). For the same line, one always finds that

$$C_0 L_0 = \varepsilon_0 \mu_0 = 1/c^2 \tag{4.60}$$

where c^2 is the light velocity squared, as could be expected because the velocity of wave propagation in the line is equal to $1/\sqrt{L_0 C_0}$.

Mutual inductance of parallel two-wire lines. Let *1* and *2* be the wires of the first line, and *3* and *4* the wires of the second line. The transverse cross section of these

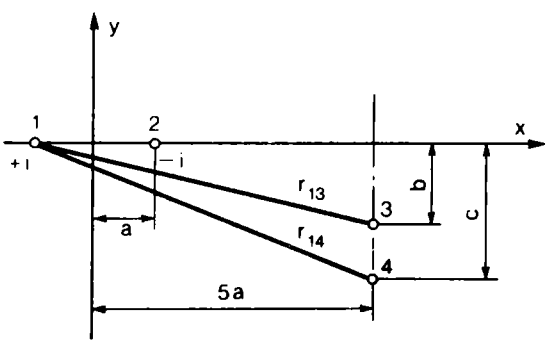


Fig. 4.13

wires is shown in Fig. 4.13. The flux of the magnetic field of the first wire that passes between the two wires of the second line is

$$\Phi_1 = (\mu_0 i/2\pi) \ln (r_{14}/r_{13})$$

where r_{mn} is the distance from wire m to wire n, as follows from (4.57) and (4.58). The flux of the second wire is expressed in a similar way. After the algebraic summation of fluxes ($\Phi = \Phi_1 - \Phi_2$) and division by the current, we obtain the mutual inductance

$$M = \Phi/i = (\mu_0/2\pi) \ln (r_{14} r_{23}/r_{13} r_{24}) \tag{4.61}$$

For the two lines of Fig. 4.13, $M = 4.6$ μH/km at $a = 1$ m, $b = 2$ m, and $c = 2.5$ m.

Mutual inductance of loops. Let $\mathbf{A}_1$ be the vector potential of current in the first loop of Fig. 4.14. By use of (2.62b), we get

$$\mathbf{A}_1 = \mu_0 \oint_{l_1} (i_1 d\mathbf{l}_1 / 4\pi r)$$

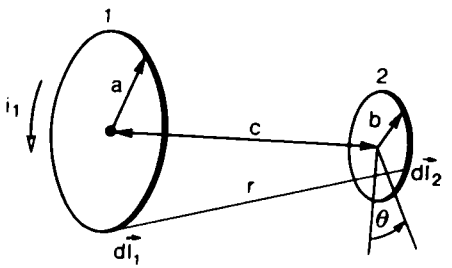


Fig. 4.14

The magnetic flux linking the second loop here is expressed by the integral of (4.58), which should be taken about the entire closed path:

$$\Phi_2 = \oint_{l_2} \mathbf{A}_1 d\mathbf{l}_2$$

Substituting the preceding expression into the latter expression and dividing by i_1 yields

$$M_{12} = M = \mu_0 \oint_{l_1} \oint_{l_2} (d\mathbf{l}_1 d\mathbf{l}_2 / 4\pi r) \tag{4.62}$$

where r is the distance from the element of the first loop, $d\mathbf{l}_1$, to the element of the second, $d\mathbf{l}_2$.

Reciprocity principle. The equality of conditions for the elements $d\mathbf{l}_1$ and $d\mathbf{l}_2$ permits us to conclude that in the general case,

$$M_{12} = M_{21} = M$$

which was proved earlier in another manner [see (4.55)].

Formula (4.62) is only applicable to vacuum or to a homogeneous medium (on multiplying by μ of the given value). The calculations by use of (4.62) always prove rather involved.

Consider the calculation of mutual inductance for two coaxial circles (loops) whose geometrical parameters are given in Fig. 4.14. For the loop *2*, the element of length is $d\mathbf{l}_2 = bd\theta$, and the scalar product is $d\mathbf{l}_1 \cdot d\mathbf{l}_2 = b \cos\theta \cdot d\mathbf{l}_1 d\theta$. The distance between the elements of two loops is

$$r = \sqrt{c^2 + a^2 + b^2 - 2ab\cos\theta}$$

Each of the elements, $d\mathbf{l}_1$, occupies the same position with respect to the set of all elements, $d\mathbf{l}_2$, of the other system. Therefore, the integral taken with respect to the elements of loop *1* is equal to $2\pi a$, and formula (4.62) assumes the form

$$M = \mu_0 ab \int_0^{2\pi} \cos\theta (c^2 + a^2 + b^2 - 2ab\cos\theta)^{-1/2} d\theta / 2$$

This is an elliptic integral that cannot be expressed as elementary functions. The

result may be written thus

$$M = \mu_0(ab)^{1/2}[(2/k - k)K - 2E/k] \tag{4.63}$$

where $k^2 = 4ab[(a + b)^2 + c^2]^{-1}$, and K and E are complete elliptic integrals given by

$$K = \int_0^{\pi/2} (1 - k^2\sin^2\alpha)^{-1/2}d\alpha$$

$$E = \int_0^{\pi/2} (1 - k^2\sin^2\alpha)^{1/2}d\alpha$$

whose values can be found in appropriate tables.

For two particular cases, these expressions take a much simpler form. If $c \gg a$ and b, then

$$M = (\mu_0\pi/16)(ab)^{1/2}k^3 \tag{4.64}$$

If $c \ll a$ and $c \ll b$ with a being approximately the same in value as b, then

$$M = \mu_0 a[\ln(8a/d) - 2] \tag{4.65}$$

where $d = \sqrt{(a - b)^2 + c^2}$ is the smallest distance between the loops.

If the loops are two coils with m and n turns respectively, the found values should be multiplied by the product mn.

Similar calculations can be made to derive the expression for self-inductance of a ring-type loop:

$$L = \mu_0 a[\ln(8a/r) - 1.75] \tag{4.66}$$

where a is the ring radius, and r is the wire radius.

The constant terms between the brackets in (4.65) and (4.66) express internal inductance. Although these expressions are in traditional use, the consideration of internal inductance presents a more complicated problem [see Sec. (8.3)].

4.3. Equations of Field Lines

The equation for **E** and **B** lines of both an electric field ($\mathbf{F} = \mathbf{E}$) and a magnetic field ($\mathbf{F} = \mathbf{B}$) may be given by

$$\mathbf{F} \times d\mathbf{l} = 0 \tag{4.67}$$

where **F** is the known field vector, and $d\mathbf{l}$ is the elementary displacement along a field line.

Expression (4.67) is nothing else but the statement of parallelism, $\mathbf{F} \parallel d\mathbf{l}$. By expressing $d\mathbf{l}$ in terms of space components (in any coordinate system) and appropriate components of the vector **F**, we arrive at a differential equation subject to the solution.

For a plane-parallel field, we obtain the following equation from (4.67), using the cartesian coordinate system:

$$F_x(x, y)dy = F_y(x, y)dx$$

or

$$dx/dy = F_x(x, y)/F_y(x, y) \tag{4.68}$$

Note that the lines of force of the magnetic field in a two-wire line coincide in geometry with the equipotential circumferences of the electric field of the same two-wire line.

The equations for field lines or flux lines of a plane-parallel field are comparatively easy to derive using the method of functions of a complex variable [(see Sec. 5.8)].

4.4. Approximate Methods of Calculation of Potential Fields

Analytic calculations often lead to complex equations, which are far from always solvable in the form of a set of certain known functions. An aproach in this case is to reduce equations to the form convenient for calculation by the methods of computing mathematics or by approximate methods. Of much practical significance are the methods of field modeling. Here we restrict ourselves to simplest approximate methods.

Graphical field mapping. For a plane-parallel field, the pattern of its field lines depends on two coordinates and obeys Laplace's equation:

$$\partial^2\varphi/\partial x^2 + \partial^2\varphi/\partial y^2 = 0 \tag{4.69}$$

($\partial\varphi/\partial z$ and $\partial^2\varphi/\partial z^2 \equiv 0$). To map the field, we first draw the cross section of two capacitor plates and then scribe "by eye" a few equipotential lines between the plates, for example,

$$\varphi_n = nU/N$$

where U is the potential difference between the plates, usually taken equal to 100 for convenience; N is the number of intervals between neighbouring equipotentials; and $n = 1, 2, \ldots$ is the order number of each equipotential line.

In Fig. 4.15 are shown the capacitor plates with potentials ranging from 0 to 100, representing the edge of a parallel-plate capacitor, and also the equipotential lines with potentials 20, 40, 60 and 80 respectively, drawn by the rule of thumb.

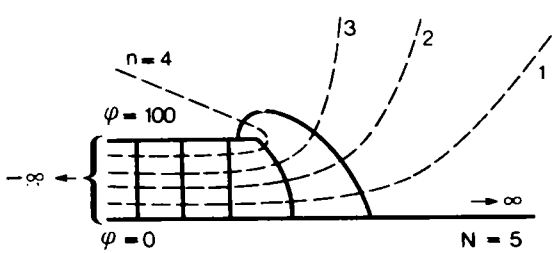


Fig. 4.15

Next, we draw orthogonal field lines in such a manner that they run at right angles to the equipotentials and the identical **E** flux flows between the neighbor lines. These lines can of necessity be drawn "by eye" too. The field map obtained certainly differs from a more accurate one given in Fig. 4.16.

After the first two operations, checking and refinement operations follow. Each field cell (Fig. 4.17) bounded by two adjacent equipotentials and two adjacent field lines (forming a curvilinear trapezoid) must display definite geometric properties. First, all the four angles should be right. Second, the ratio b/h must be constant throughout the extent of any "tube" of force, being bounded by adjacent field lines. Here b is the average width of a trapezoid, and h is its average height. The second

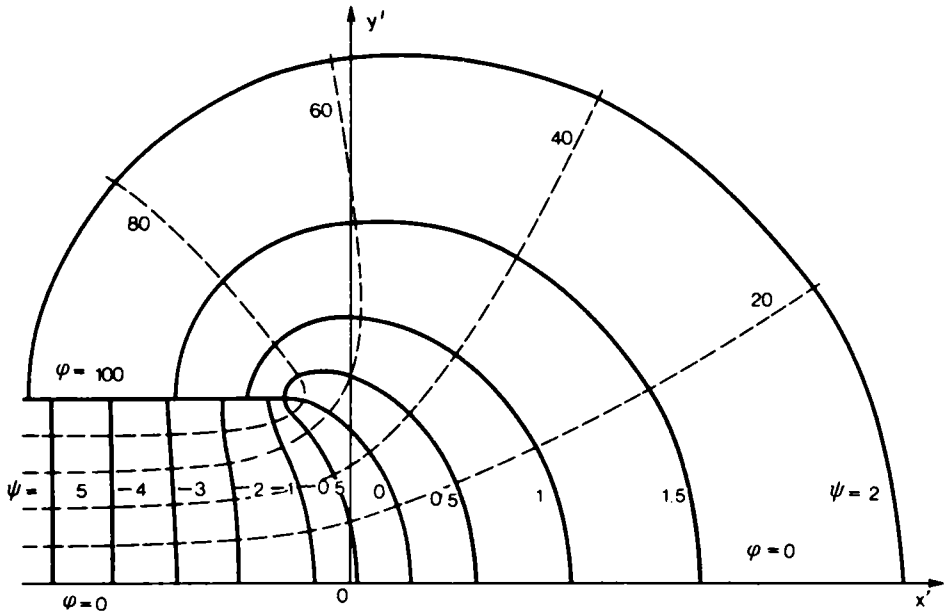


Fig. 4.16

condition stems from the fact that the average field strength between two equipotentials is in inverse proportion to the spacing between them

$$E_{av} \sim 1/h$$

while the E flux between neighbor field lines must remain constant. This flux is proportional to the average trapezoid width and average field strength:

$$\Psi_0 \sim E_{av} b \sim b/h = \text{constant} \tag{4.70}$$

The sign $\sim$ in the two latter formulas implies proportionality.

Of use can be checking the pattern construction by the grid rule (Fig. 4.18). In the

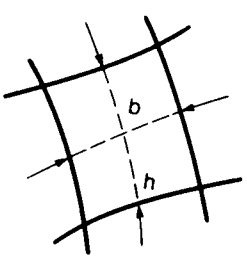


Fig. 4.17

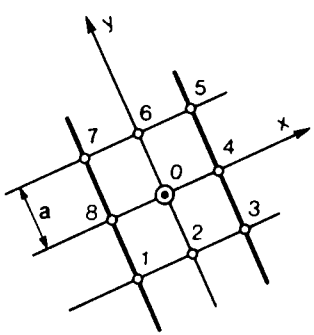


Fig. 4.18

network having orthogonal meshes (cells) each with a side equal to a, the potential of the center point is equal to the average value of potential at surrounding points. Really, Laplace's differential equation may approximately be expressed as a finite difference

$$\partial^2\varphi/\partial x^2 \approx (\varphi_4 - \varphi_0)/a^2 - (\varphi_0 - \varphi_8)/a^2 \tag{4.71}$$

Here the numerator gives the difference between the first derivatives, as related to a unit step (interval), i.e. approximately expresses the second derivative along the axis *804* taken as the x axis. On performing a simple cancellation, we get

$$a^2(\partial^2\varphi/\partial x^2) \approx \varphi_4 + \varphi_8 - 2\varphi_0$$

Taking the line *206* as the y axis we get

$$a^2(\partial^2\varphi/\partial y^2) \approx \varphi_6 + \varphi_2 - 2\varphi_0$$

Summing up the two latter equations and applying (4.69) gives

$$\varphi_0 = (\varphi_2 + \varphi_4 + \varphi_6 + \varphi_8)/4 \tag{4.72}$$

Using the lines *105* and *307* as the x and y axes, we obtain a similar result:

$$\varphi_0 = (\varphi_1 + \varphi_3 + \varphi_5 + \varphi_7)/4 \tag{4.73}$$

The higher the skill in constructing field lines, the easier it becomes to map the field that closely resembles the real one. For this it is very useful to drill oneself in constructing the field pattern for cases amenable to an analytic solution.

Example 4.1. Construct equipotentials and flux lines for the field of a parallel-plate capacitor in which one plate is an infinite plane $y = 0$ and the other is an infinite half-plane. The cross section of the plates of such a capacitor is presented in Fig. 4.19.

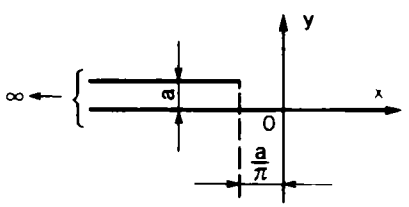


Fig. 4.19

In parametric form, the equation of equipotentials, φ = constant, and of flux lines (field lines), Ψ = constant, was given by Maxwell and written as

$$\begin{aligned} x &= (a/\pi)(\Psi + e^{\Psi}\cos\varphi) \\ y &= (a/\pi)(\varphi + e^{\Psi}\sin\varphi) \end{aligned} \tag{a}$$

On the strength of these equations we should draw the lines of potential in a step of $0.2U$, where U is the total potential difference between the plates, and also scribe force lines at an interval of $-1 \leqslant \Psi \leqslant 2$ so that Ψ can change by 0.5 between the neighbor lines and by 1 in the interval $-5 \leqslant \Psi \leqslant -1$.

Prior to building up the pattern (more accurately, to be in a position to start with the construction), we need to answer the following questions.

1. What are the upper plate coordinates that correspond to-equations (a)?
2. What are the plate potentials and the potential difference between the plates?

Solution. 1. The y coordinate of the upper plate cannot depend on the flux, Ψ, and hence its potential should be taken equal to $\varphi = \pi$. Obviously, in this case $y = a$.

Knowing the potential on the upper plate, we write the first of the two equations for this plate in the form

$$x = (a/\pi)(\Psi - e^{\Psi}) \tag{b}$$

because $\cos\varphi = \cos\pi = -1$. It is apparent that the x coordinate of the upper plate depends on the flux, i.e. on Ψ; at $\Psi = -\infty$, it is clear that $x = -\infty$. It now remains to determine the maximum possible value of x that satisfies equation (b), which is just the coordinate of the upper plate edge.

Employing the classical method of determining a maximum, we seek the solution to the equation

$$\partial x/\partial\Psi = (a/\pi)(1 - e^{\Psi}) = 0$$

The solution is obvious: $\Psi = 0$. The respective maximum value is

$$x = -a/\pi$$

2. We have determined the potential on the upper plate in analyzing the first question; namely, $\varphi = \pi$. The same considerations force us to believe that the potential on the lower plate is $\varphi = 0$.

The field pattern so constructed is given in Fig. 4.16.

Plane-meridian field mapping. This type of construction calls for meeting certain additional requirements. The field of electrodes in the form of bodies of revolution is a typical example of such a field. In the cylindrical system of coordinates, the axis of which coincides with axis of rotation of electrodes, the pattern of such a field is independent of the angular coordinate, α. In any plane, α = constant, the field patterns are identical and are only dependent on two coordinates, r and z. Fig. 4.20 illustrates the field formed between a point and a disk.

After mapping the lines in an approximate manner, one should check the map and make it more accurate using now a new rule as a guide: inside of each section (Fig. 4.21) of the field map, the parameters b, h, and r must satisfy the equality

$$br/h = \text{constant} \tag{4.74}$$

Really, now the tube of flux Ψ_0 has a thickness (in the direction of the normal to the plane of the paper) which depends on the distance from the observation point to

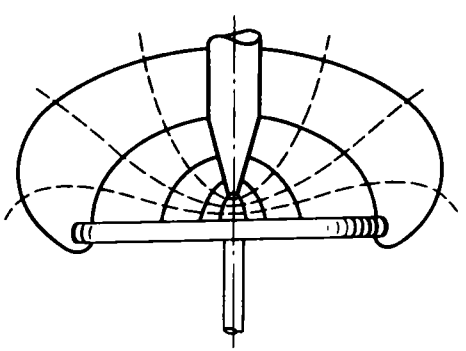


Fig. 4.20

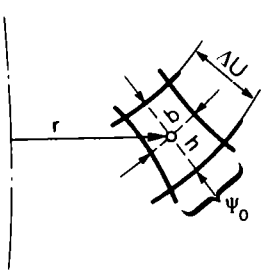


Fig. 4.21

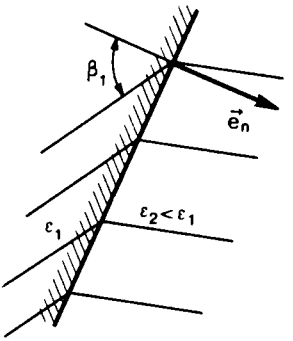


Fig. 4.22

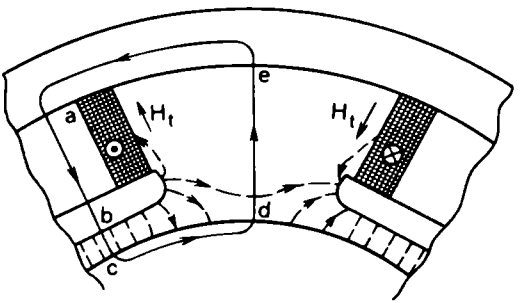


Fig. 4.23

the axis: this thickness is proportional to the radius.

Throughout the present book we will repeatedly analyze plane-meridian fields, to which belong the field of a dipole, the magnetic field of a coil, and the field of a spheroid polarized along the axis of symmetry.

Additional conditions for field mapping in the presence of differently polarizable media. If a boundary between the dielectrics of permittivities ε_1 and ε_2 exists within the field, then the field mapping requires fulfilling the boundary conditions. In this case, on passing from one medium to the other, the angles β_1 and β_2 at which field lines cross the interface obey Eq. (3.30):

$$\tan\beta_1/\tan\beta_2 = \varepsilon_1/\varepsilon_2 = E_{2n}/E_{1n}$$

Then, according to (3.26),

$$D_{1n} = D_{2n}$$

It is of interest to compare Figs. 4.22 and 3.9.

Determination of capacitance. Using the field map, we can determine the capacitance between two electrodes. The density of electric flux going from one electrode to the other is proportional to the number of tubes of flux (in a plane-parallel field). The potential difference is known from the conditions of field pattern construction and enters into the scale constant for each field cell. Therefore, the ratio between the number of tubes leaving the electrodes and the potential difference is proportional to the capacitance.

Magnetic field mapping. In the region where currents are not present (curl $\mathbf{H} = 0$), it is possible to express the strength (or induction) of a magnetic field as the gradient of magnetic potential, $\mathbf{H} = -\operatorname{grad}\varphi_m$, and to use Laplace's equation. In considering the field in an air gap, the surfaces of ferromagnetic electrodes may be assumed equipotential (if the permeability is high). The air gap between the rotor of an electric machine and its poles satisfies this condition to a high degree of approximation. A section of the electric machine is shown schematically in Fig. 4.23.

The integral of the magnetic field strength along the closed path a, b, c, d, e must be equal to the magnetomotive force of the first core winding. If the permeabilities of the ferromagnetic cores, rotor, and stator are high, the magnetic field strength can be taken equal to zero.

In this case the integral around the closed loop reduces to the integral taken between b and c

$$\oint \mathbf{H} d\mathbf{l} = \int_b^c H dl \tag{4.75}$$

if the windings are symmetric about the poles. With the latter condition being met, the line dl is always normal to the lines of an insignificant portion of the magnetic flux that flows past the rotor.

But in the air, according to the condition, $\mathbf{H} = -\text{grad}\,\varphi_m$. We then may assume that the difference in magnetic potentials between the pole piece and rotor is

$$\varphi_b - \varphi_c = \theta_1 \tag{4.76}$$

where $\theta_1 = i_1 w_1$ is the magnetomotive force of the winding.

If the longitudinal dimensions of the rotor and poles are large as against the gap between the poles and rotor, all the conditions for a magnetic field do not differ from the conditions for a plane-parallel electric field. But this is not so for the region occupied by current-carrying windings. This region should be left out of the consideration; it may then be assumed that a longitudinal component, $H_t = iw/l$, exists on the surface of windings.

4.5. Simple Approximate Calculations of Capacitance and Inductance

Method of average potentials of an electric field. To start with the calculations, we replace the actual constancy of potential on an electrode by the assumption of constancy of a charge density (surface charge density ξ or line charge density τ), considering that electrodes carry charges $\pm q$. A hypothetical potential distribution corresponding to the assumed charge distribution is found from simple formulas analogous to (1.21b):

$$\varphi = \xi \int (dS/4\pi\varepsilon_0 R) \text{ or } \varphi = \tau \int (dl/4\pi\varepsilon_0 R) \tag{4.77}$$

Next we find an average potential on electrodes, which is supposed to be equal to the real one.

Actually, the electrode potential remains constant and the potentials of (4.77), which correspond to the constant-density charge distribution, level off as a result of the additional distribution of a self-charge equal on the average to zero. We will not dwell on how to specify and substantiate this approximate method suggested by an American engineer Howe, but only note that it generally gives quite satisfactory results.

Assume two wires of lengths l_1 and l_2 carry charges q_1 and q_2. The potential at any point on the first wire in the field of the second uniformly distributed charge will then be given by

$$\varphi'_{1,2} = \tau_2 \int_{l_2} (dl_2/4\pi\varepsilon_0 R) \tag{4.78}$$

where R is the distance from the element dl_2 to the observation point on the first element dl_1; $\tau_2 = q_2/l_2$.

The average potential on the first wire is

$$\varphi_{1,2} = (1/l_1) \int_{l_1} \varphi'_{1,2} dl_1 = (q_2/l_1 l_2) \int_{l_1} \int_{l_2} (dl_1 dl_2/4\pi\varepsilon_0 R) \tag{4.79}$$

By comparing (4.79) with the known system of equations (4.22), we can state that

$$\alpha_{12} = (1/l_1 l_2)\int_{l_1}\int_{l_2}(dl_1 dl_2/4\pi\varepsilon_0 R) \tag{4.80}$$

The average potential of the first wire due to its self-charge is found in a similar way, and therefore the coefficient α_{11} is expressed as

$$\alpha_{11} = (1/l_1^2)\int\int(dl'_1 dl''_1/4\pi\varepsilon_0 R) \tag{4.81}$$

Here R is the distance between various elements dl' and dl'' of the first wire. The assumption here is that the charge, $\tau_1 = q_1/l_1$, distributes itself on the axis of the wire forming the first electrode (Fig. 4.24). If the wire is round in cross section, the smallest distance between the elements is $R = r_0$, where r_0 is the wire radius.

Knowing the coefficients α_{mn}, it is an easy matter to estimate the capacitance between the pair of wires. For two wires having $\alpha_{12} = \alpha_{21}$, $q_1 = q$, and $q_2 = -q$, we find from (4.80):

$$\varphi_1 - \varphi_2 = U = (\alpha_{11} - \alpha_{21})q - (\alpha_{12} - \alpha_{22})q$$

and

$$C = q/U = 1/(\alpha_{11} + \alpha_{22} - 2\alpha_{12}) \tag{4.82}$$

Inductance calculation by the approximate method of lengths. In expression (4.62) for M, we can replace the closed-loop integrals by the sum of integrals, each extended along a small path amenable to simple computation. Formula (4.62) can thus be rewritten to assume the form

$$M = \mu_0 \sum_{k=1}^{m} \sum_{p=1}^{n} \int_{l_{1k}} \int_{l_{2p}} (d\mathbf{l}_1 d\mathbf{l}_2/4\pi R) \tag{4.83}$$

Here the first loop is broken down into m length elements, and the second into n length elements. In determining self-inductance, a companion equation applies:

$$L_{11} = \mu_0 \sum_{k=1}^{m} \sum_{p=1}^{n} \int_{l_{1k}} \int_{l_{1p}} (d\mathbf{l}''_1 d\mathbf{l}'_1/4\pi R) \tag{4.84}$$

where R is the distance between the two elements of the same wire. As in the case of determining α_{11}, the smallest distance between the two elements here is equal to the wire radius (see Fig. 4.24).

If the loop lengths are straight paths, we can replace the scalar product by the product of absolute quantities, taking the constant cosine of the angle outside the integral. In this case, comparing (4.83) and (4.84) with (4.80) and (4.81), we get

$$M = \alpha_{12} l_1 l_2 \mu_0 \varepsilon_0 \cos\theta \tag{4.85}$$

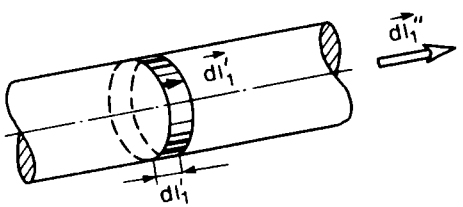


Fig. 4.24

where θ is the angle formed between the lengths $d\mathbf{l}_1$ and $d\mathbf{l}_2$.

The angle θ is different for different terms when taking the summation with respect to individual lengths. Formula (4.85) was derived by a Soviet scientist L. A. Tseytlin. He also introduced the notion of the inductance of lengths into calculation practice.

The calculations by the formulas given above are particularly convenient if the coefficients α_{12} are known. For simple cases, the computations do not present difficulties.

4.6. Potential Field Modeling. Electrolytic Tanks

Electrolytic tank. In studying various fields not amenable to simple calculations, it is usual to model the fields using an electrolytic tank, i.e. the bath with an electrolyte. The primary requirement here is to ensure the geometric similarity between the real field and its model.

To convert the data such as U' and R' obtained from a model to the data such as U and R expected to be adequate for a real field, use is made of scale factors, or the coefficients of model equations.

A conductor of rectangular cross section (sides a and b) and of length l has a conductivity σ and resistance $R = l/ab\sigma$. A geometrically similar model of the conductor in a reduced scale, $l/l' = a/a' = b/b' = m_l$, $\sigma' = \sigma/m_\sigma$, has a resistance

$$R' = l'/a'b'\sigma' = (l/ab\sigma)m_l m_\sigma$$

The scale factor m_R for the resistance R is then given by

$$R = R' m_R, m_R = 1/m_l m_\sigma \tag{4.86}$$

For a hemispheric well-conducting electrode embedded in the earth of conductivity σ (Fig. 4.25) at a very long distance from a second electrode, the leakage resistance is

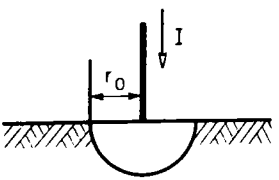


Fig. 4.25

found in such a manner. The field strength about the first electrode is

$$E = E_R = I/2\pi r^2\sigma$$

where I is the current leaking off the electrode. The potential difference between the surface of the electrode of radius r_0 and the second electrode is given by

$$U = \varphi_1 - \varphi_\infty = \int_{r=r_0}^{r=\infty} E_R dr = I/2\pi\sigma r_0$$

When making measurements in an electrolytic tank, the resistance of a similar electrode is expressed as

$$R' = U'/I' = 1/2\pi\sigma' r'_0 = m_\sigma m_l/2\pi\sigma r_0 = R/m_R$$

at $m_R = R/R' = 1/m_\sigma m_l$.

Protective grounding and step voltage. Grounding, i.e. connection of parts of electrical equipment to grounded electrodes, is laid out to protect the attending personnel from the hazards of high voltages. Its purpose is to keep the contact voltage between two exposed points from rising to the values dangerous to health. A man approaching a ground rod with a step of 0.7 m is subject to a voltage that exists between the points (on the earth surface) lying at a distance of 0.7 m from each other. This voltage is referred to as a step voltage.

Example 4.2. A geometrically similar model of a grounding structure, made at a scale of 1 to 50, is immersed in an electrolytic tank with $\sigma = 1.5 \times 10^{-2}$ S/cm. The real grounding system consists of a number of tubes 8 cm in diameter driven into the ground to a depth of 2 m along the perimeter of a rectangle. From the consideration of symmetry, under test is only one fourth of the model of the grounding system (Fig. 4.26*a*). A second, "far remote" electrode is represented by a copper plate disposed on the opposite curved wall of the tank. The tank radius R is considered large, as is the tank depth, in comparison with the linear dimensions of the grounding ($R \approx 5l'$ to $10\,l'$). The dashed line d' in Fig. 4.26*a* is the direction of the quickest change in potential φ' at $I' = 3$ A. The drop in φ' is found experimentally and shown in Fig. 4.26*b*.

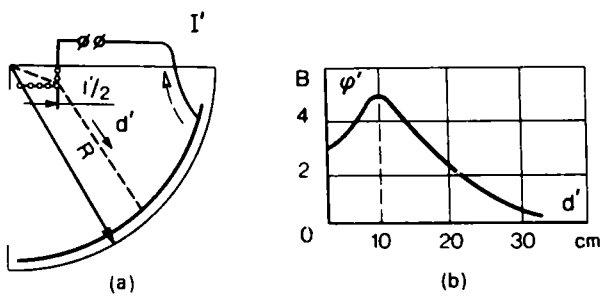


Fig. 4.26

Plot the φ-d curve for the real grounding system at a fault-to-ground current I of 1 000 A and ground conductivity σ of 10^{-4} S/cm.

Solution. The problem comes to determining the model equation coefficients or scale factors for the potential and the coordinate, given in the equalities $\varphi = \varphi' m_\varphi$ and $d = d' m_d$.

Using simple relationships

$$U = EL = Jl/\sigma = Il/S\sigma$$

we readily get

$$U = I' m_I \cdot l' m_l / S' m_S \cdot \sigma' m_\sigma = U' m_U$$

where

$$I' m_I = I, \ldots, \sigma' m_\sigma = \sigma \text{ and } U' = I' l'/S'\sigma$$

Hence, the scale of voltage or potential is

$$m_\varphi = m_U = m_I m_l / m_l^2 m_\sigma$$

($m_S = m_l^2$). For the concrete values of the parameters given in this example, $m_\varphi = 1\,000$.

Knowing the scale factor in addition to the given geometric scale of the model, we can construct the graph, therewith changing the scale division along the vertical and the horizontal on the graph of Fig. 4.2*b*.

By measuring the potential difference at a step of 0.7 m on the curve of Fig. 4.26*b*, it is easy to plot the graph of step voltage (in the direction *d*) against the coordinate and verify whether or not the step voltage at every step is within the limits specified by safety rules ($U \leqslant 40$ V). If the step voltage exceeds the permissible value, it is necessary to smooth out the rate of drop in potential by changing the configuration of grounding connections, lay out insulating mats, or guard the hazardous zones (commonly lying in close vicinity to ground rods and their protruding parts), or take appropriate precautions against high short-circuit currents dissipating into the earth.

Conductivity and resistance. In the simplest case, we can represent a mixture of two heterogeneous materials by a number of balls located at the sites of a cubic lattice. Designate the ball conductivity as σ_b and the conductivity of the base as σ_0. If $\sigma_b \gg \sigma_0$, then it is possible to model the mixture (structure) by placing an electrode in the form of one-eighth part of the ball in one of the corners of a cubic bath (Fig. 4.27). The two walls *1* and *3* are made conducting, of which one is adjacent to the electrode. The remaining four walls do not conduct current ($J_n = 0$). In accordance with the conditions of symmetry, the appropriately disposed planes in the system of an unbounded cubic lattice display exactly the properties mentioned above, i.e. they are equipotential and parallel to the lines of current.

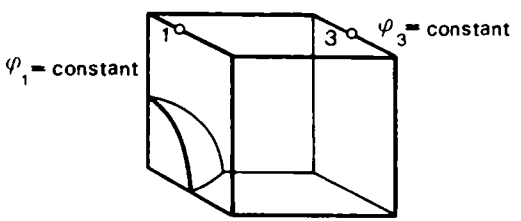


Fig. 4.27

The resistance between the conducting walls of the model is

$$R' = U'/I' = |E'|_{av} a'/J_{av}(a')^2 \tag{4.87}$$

where a' is the length of the model (cube) edge.

The resistance R' is equal to the actual resistance divided by the appropriate scale factor given by (4.86).

The structure (intrinsic) conductivity may be expressed as

$$|E|_{av}/J_{av} = Ra = \sigma_{str}$$

where a is the length of the cubic lattice edge.

The structure conductivity defined for the model is

$$\sigma'_{str} = \sigma_{str}/m_\sigma \tag{4.88}$$

If the structure contains nonconducting ball inclusions ($\sigma_b = 0$) in the medium of conductivity σ_0, then R and σ_{str} will substantially be different from those of the structure described above. But the scale factor will remain the same

provided the conditions of geometric similarity are observed. In the system of Fig. 4.27, conducting one-eighth ball parts should now be replaced by nonconducting ones, with the walls *1* and *3* left conducting as before.

Coefficients of capacitance and potential. Fig. 4.28 shows a measuring curcuit for determining the coefficient α_{12} in a three-phase cable. The setup has electrodes geometrically similar to the stranded conductors. The electrodes are immersed into an electrolyte that fills a metal cylinder in the form of a cable sheath. The procedure is as follows. When the position of wiper *2* is such that the null inducator *0* shows zero, the readings U_2 and I_1 are taken from voltmeter V_2 and ammeter A_1. The ratio U_2/I_1 multiplied by the scale factor gives the coefficient of potential:

$$\alpha_{21} = (U_2/I_1)m_\alpha$$
$$m_\alpha = (l'/l)(\sigma/\varepsilon\varepsilon_0) \tag{4.89}$$

where σ is the electrolyte conductivity; ε is the permittivity of cable insulation; l'/l is the ratio of the axial model length (tank depth) to the cable length l being defined when determining the charge density per unit length, $\tau = q/l$. The geometric similarity is assumed to be observed in the transverse cross section.

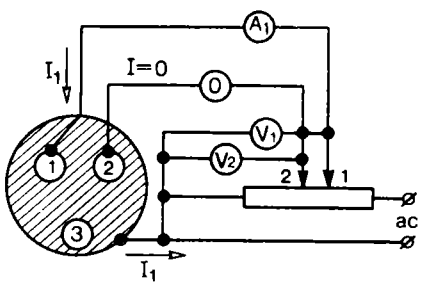


Fig. 4.28

Knowing the reading U_1 of voltmeter V_1, it is easy to determine the other coefficient of potential:

$$\alpha_{11} = (U_1/I_1)m_\alpha \tag{4.90}$$

Flat tanks and conducting sheets. In the case of plane-parallel fields, where the potential depends only on two cartesian coordinates, $\varphi(x, y)$, modeling is made using flat tanks. The tank is a thin layer of electrolyte with electrodes whose generating lines pass through the entire depth of electrolyte to the nonconducting bottom. Another approach to field modeling is to use paper impregnated with a suitable conducting compound or uniformly coated with the thinnest layer of graphite or some other conducting substance. When employing conducting sheets for modeling, the conducting bodies of requisite configuration, tightly pressed against the sheet, serve as electrodes.

The fringe effect of the tank or sheet can in essence be eliminated by elaborating a little the tank design.

4.7. The Reciprocity Principle for a DC Circuit

The methods of field theory offer the possibility of establishing certain general statements which are of significance in circuit theory too. The principle of reciprocity applied to a linear electric circuit is one of the statements.

Figure 4.29 illustrates a linear passive two-port that satisfies the reciprocity principle expressed in the form

$$I_1'U_1'' + I_2'U_2'' = I_1''U_1' + I_2''U_2' \tag{4.91}$$

where primed and double-primed symbols denote the quantities typical of certain (generally, any) two conditions. It is only the external circuit that determines the difference in conditions because the four-terminal network is passive and its properties are invariable. However, whatever the condition that may exist, the current entering the pole a is equal to the current leaving the pole c.

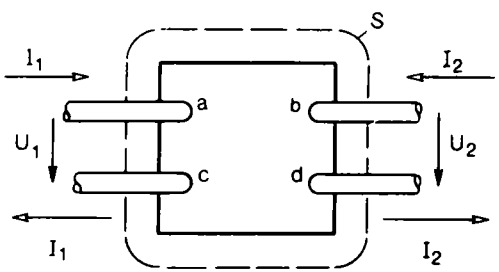


Fig. 4.29

The voltages are defined as the respective potential differences:

$$U_1' = \varphi_a' - \varphi_c', \quad U_1'' = \varphi_a'' - \varphi_c''$$
$$U_2' = \varphi_b' - \varphi_d', \quad U_2'' = \varphi_b'' - \varphi_d''$$

Let us take the integral of the artificially derived difference over the closed surface S:

$$\oint (\mathbf{J}''\varphi'' - \mathbf{J}''\varphi')d\mathbf{S} = \int [\operatorname{div}(\mathbf{J}'\varphi'') - \operatorname{div}(\mathbf{J}''\varphi')]dV = 0 \tag{4.92}$$

which is transformed in accordance with Gauss' theorem. The volume integral that results is identically equal to zero, which is easy to prove. Really, taking the divergence of the product, we get

$$\operatorname{div}(\mathbf{J}'\varphi'') = \varphi'' \operatorname{div}\mathbf{J}' + \mathbf{J}' \operatorname{grad}\varphi'' = 0 - \mathbf{J}'E'' = -\mathbf{J}'\mathbf{J}''/\sigma \tag{4.93}$$

Equations (4.92) and (4.93) take account of the fact that $\operatorname{div}\mathbf{J} = 0$ (Kirchhoff's current law). The subtrahend in (4.92) also proves equal to the same quantity given in (4.93):

$$\operatorname{div}(\mathbf{J}''\varphi') = -\mathbf{J}''\mathbf{J}'/\sigma$$

Invoking again Eq. (4.92) and integrating over the closed surface gives Eq. (4.91). Indeed, the current density is nonzero only

where the surface S traverses one of the four wires. Where the surface crosses any wire, the potential may be considered constant. Therefore, for the two crossed wires a and c the integral of the first term can be written as

$$\int_{S_1} \mathbf{J}' \varphi'' d\mathbf{S} = -I_1' \varphi_a'' + I_1' \varphi_c'' = -I_1' U_1'' \tag{4.94}$$

where S_1 is a part of surface S crossing only the wires a and c. For the wire a crossed by the surface, the integration of current density gives I_1' negative in sign because, according to the condition, I_1' is the current entering the network along the wire a. But $\mathbf{J}' d\mathbf{S}$ is positive when the current leaves the surface S (by the adopted rule, the direction of $d\mathbf{S}$ is out of the paper).

Integrating both terms over the entire closed surface, we arrive at (4.91), thereby constructing the proof of this formula.

The applicability of the reciprocity principle for any linear ac circuit without revolving elements follows from the reciprocity principle applied to the coefficients of capacitance and potential, $\beta_{ik} = \beta_{ki}$ and $\alpha_{ik} = \alpha_{ki}$, and also to the coefficients of mutual inductance, $M_{ik} = M_{ki}$.

4.8. Theorems of a Magnetic Flux and Induced Electric Charge

Application of the principle of reciprocity. Many electrical devices depend for their operation on the emergence of an emf with changes of a magnetic flux linking a certain conducting loop. It is sometimes easier to determine this flux by using the principle of reciprocity if we then can restrict ourselves to the calculation of a simpler field.

An instance of the circuit in Fig. 4.30 may serve as an illustrative example, though hardly real from the technical standpoint. We have to determine the emf $\varepsilon = u_2$ induced in the turns of a solenoid diametrically pierced by an insulated wire having a voltage source u_1 connected across it. It is very easy to calculate the flux that links the straight wire when the field is due to the solenoid current. But it is more difficult to find the flux that links the multiturn solenoid from the magnetic field of the wire carrying the current i_1.

The above statement that refers to the possibility of simplifying the calculations by use of the principle of reciprocity is of particular importance when the problem at hand is to determine the loop-linking flux due to the magnetization of a ferromagnet.

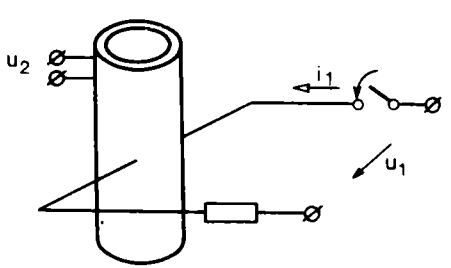


Fig. 4.30

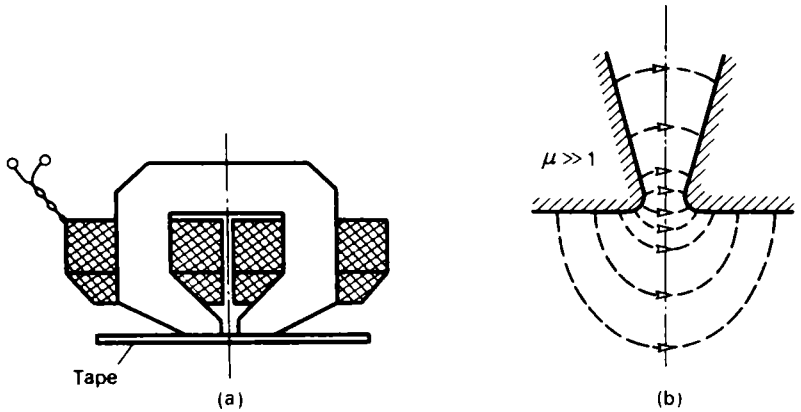


Fig. 4.31

Consider an example of the changing flux that links the coil of a reading head (Fig. 4.31*a*) as the tape carrying the recorded information slides past the head. To determine the flux linkage, it is possible (but not advisable from the author's viewpoint) to proceed from the calculation of the magnetic field of the tape. Instead, it is more advantageous to apply the principle of reciprocity in the calculation and then evaluate the flux linkage in a much simpler way and much more accurately. To make this calculation on the basis of the magnetic flux theorem presented below, it is enough to know the magnetic field in the tape region that results from the unit current in the head winding (Fig. 4.31*b*). With a small gap between the poles, the field decays very fast. Moreover, the field can be calculated or modeled in a rather simple way.

The calculation of a charge induced on the electrodes by a polarized dielectric is similar to the calculation of a flux linkage due to magnetization. Both methods of calculation use the analogous theorems, which are one of the corollaries to the principle of reciprocity.

Flux linkage theorem and its application. The sought-for flux that links a given loop and results from the magnetization of a body can always be found from the formula

$$\Psi = \mu_0 I' \int_V \mathbf{H}'_I \mathbf{M} dV \tag{4.95a}$$

where $\mathbf{H}'_I$ is the field strength of the loop under examination when the current I' flows through it (the effect of the polarizable body on the field $\mathbf{H}'_I$ being disregarded); $\mathbf{M}$ is the magnetization of the body in question, inside the volume element dV; and V is the volume of the magnetized body being considered.

The formula of flux linkage becomes particularly simple if both $\mathbf{M}$ and $\mathbf{H}'_I$ are constant within the body volume V:

$$\Psi = (\mathbf{H}'_I / I') \mu_0 \mathbf{M} V \tag{4.95b}$$

Example 4.3. Inside a long multiturn solenoid having $n_0 = 5$ turns in 1 cm of length there is a thin ferromagnetic film disposed parallel to the solenoid axis. The film has the shape

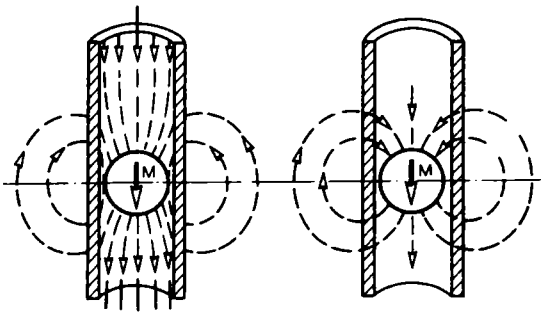


Fig. 4.32

of a disk with a thickness $\tau = 2\,000\,\text{Å} = 2 \times 10^{-5}$ cm and a radius $r = 0.5$ cm (Fig. 4.32). What is the emf induced at the solenoid terminals if the film magnetization component parallel to the solenoid axis varies uniformly from $\mu_0 M = 1$ T to $\mu_0 M = -1$ T in a period $T = 10$ ns $= 10^{-8}$ s?

Solution. By formula (4.95b), the magnetic flux linkage

$$\Psi = (H_I'/I')\mu_0 MV = n_0 \mu_0 M \tau \pi r^2$$

In the course of change of magnetization, the emf

$$\mathscr{E} = -d\Psi/dt = 2\Psi_0/T = 1.57 \text{ V}$$

where $\Psi_0 = \Psi$ at $\mu_0 M = 1$ T.

Note. Fig. 4.32*a* shows the solenoid field with a magnetized body inside the solenoid. It would certainly be possible to determine $\Delta\Psi$ directly, too, by calculating the additional flux linking the loop (Fig. 4.32*b*). But this involves complex computations.

Foner magnetometer. In recent years this device has gained wide recognition in the investigation of ferromagnets. The principle of the magnetometer is easy to explain using the theorem under study. The device is adopted to measure the magnetization of a specimen — a ball of radius ρ — placed between electromagnet poles in a field $H_0 = H_x$ (Fig. 4.33*a*). It should be remembered that the magnetization of many materials, $M = M_x$, is very low in comparison with the strength H_0. For example, $M/H_0 = 0.016$ in the Gaussian system. These conditions make the use of a number of known methods completely impracticable. In the magnetometer of Fig. 4.33, the specimen connected to a vibrating rod executes low-value oscillations in the direction parallel to the axis of measuring coils and perpendicular to the electromagnet field ($v = v_y$ at $H_0 = H_x$). It turns out that the emf induced in the coils is proportional to the magnetization of the specimen, its volume V, and the rate of its motion, v:

$$\mathscr{E} = kM_x Vv \tag{4.96}$$

at $M = M_x$.

In Fig. 4.33*b* is illustrated the pattern of a magnetic field of two parallel coils, with the current in one coil flowing in the direction opposite to that in the other. With the system of coordinates being symmetric, it is obvious from the symmetry

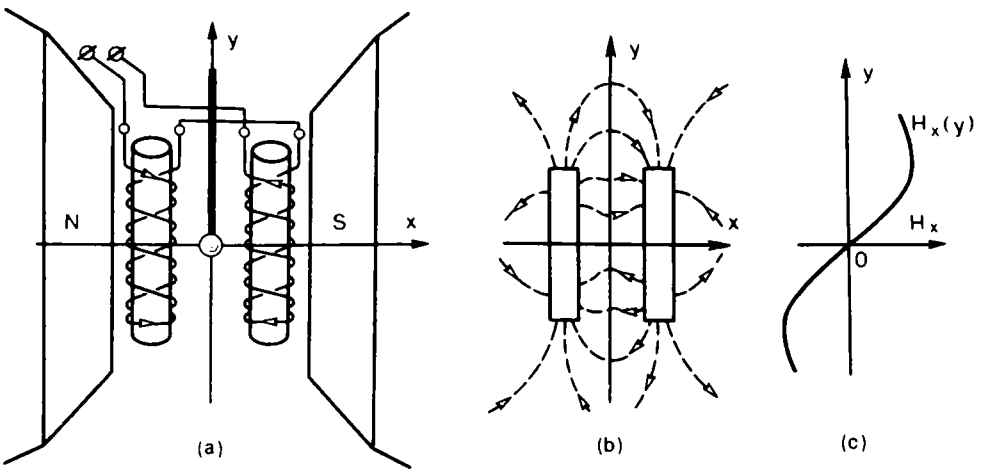


Fig. 4.33

considerations that only the xth field strength component exists on the y axis ($x = 0$, $z = 0$). This component is the odd function of y. At low values of y, it can be assumed that

$$H^0 = H_x^0 = ky \tag{4.97}$$

where $H^0 = H_I'/I'$. The H_x-y relationship appears in Fig. 4.33*c*.

The results of the calculation or measurement can be employed to evaluate the coefficient k. In this case, by use of (4.96) we find the flux linkage between the coils due to the ball magnetization:

$$\Psi = H^0 M_x V = kM_x V y \tag{4.98}$$

With the specimen set in vibration,

$$\mathscr{E} = -\partial\Psi/\partial t = -kM_x V \partial y/\partial t \tag{4.99}$$

But $\partial y/\partial t = v$, and therefore (4.99) is in agreement with (4.96).

Proof of the magnetic flux theorem. Visualize a volume element dV taken off a magnetized body (Fig. 4.34). Assume it has the shape of a pill box with a generating line $d\mathbf{l} \parallel \mathbf{M}$ and sides dS, whose planes are normal to the magnetization vector (i.e.

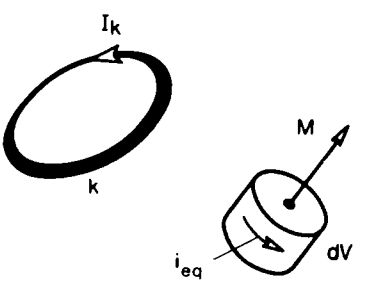


Fig. 4.34

$d\mathbf{S} \parallel \mathbf{M}$), so that

$$dV = dl dS = dl dS$$

The magnetic moment of the element $d\mathbf{m}$ under examination can be represented by an equivalent loop of current i_{eq} that encloses an area $d\mathbf{S}$:

$$d\mathbf{m} = \mathbf{M}dV = i_{eq}d\mathbf{S}$$

From the two latter equations it follows that

$$\mathbf{M}dl = Mdl = i_{eq} \tag{a}$$

The flux Ψ passing through the macroscopic loop k contains an elementary component $d\Psi$ that owes its existence to the magnetized element $\mathbf{M}dV$. This component can be expressed as an equality

$$d\Psi = i_{eq}dL_M \tag{b}$$

where i_{eq} is the equivalent loop current; and dL_M is the mutual inductance linking the equivalent loop and the given loop k.

Based on the principle of reciprocity we can define dL_M in terms of the flux through the area $d\mathbf{S}$ at a unit current I_k in the given macroscopic loop k. So,

$$dL_M = \mu_0 \mathbf{H}_k' d\mathbf{S}/I_k' \tag{c}$$

where $\mathbf{H}_k'$ is the field strength vector due to I_k' in the loop k. The vector $\mathbf{H}_k'$ is defined with respect to the location of a particular element $d\mathbf{S}$.

Substituting (a) and (c) into (b) gives

$$d\Psi = \mu_0 \mathbf{H}_k'(\mathbf{M}dl)d\mathbf{S}/I_k'$$

or, since the condition is that $\mathbf{M} \parallel d\mathbf{S}$ and $dl dS = dV$,

$$d\Psi = (\mu_0/I_k')\mathbf{H}_k'\mathbf{M}dV$$

On integrating through the volume V for which $\mathbf{M}\mathbf{H}_k' \neq 0$, we arrive at expression (4.95), which was to be proved.

Induced charge theorem. This theorem is analogous to the theorem described above.

If polarization $\mathbf{P}$ appears between the plates of a short-circuited capacitor (Fig. 4.35*a*), then a charge flows through the path short-circuiting the plates:

$$\Delta Q = -(\int \mathbf{E}'\mathbf{P}dV)/U' \tag{4.100}$$

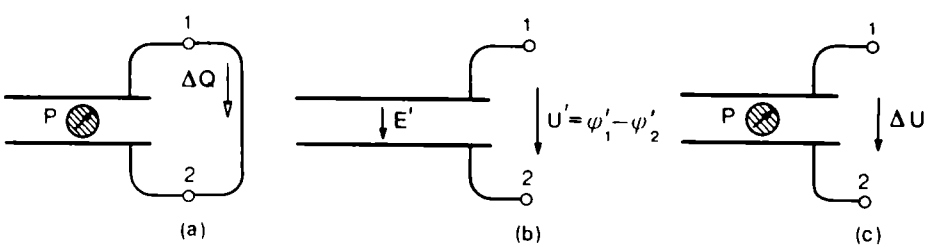


Fig. 4.35

Here $\mathbf{E}'$ is the electric field strength of the capacitor at U' across it in the absence of a polarized body (Fig. 4.35*b*); and V is the volume of the polarized body.

In formula (4.100), $\Delta Q > 0$ when the charge passes from plate *1* to plate *2*; the voltage in (4.100) is then defined as the difference:

$$U' = \varphi_1' - \varphi_2' \tag{4.101}$$

As with the theorem of magnetic flux, expression (4.100) becomes much simpler if $\mathbf{E}'\mathbf{P}$ remains constant throughout the body bulk.

A similar theorem can readily be formulated for insulated plates (Fig. 4.35*c*) where $\Delta Q = 0$. With a polarized body inserted in the capacitor, the voltage between the plates rises by

$$\Delta U = -(\int \mathbf{E}'\mathbf{P}dV)/CU' \tag{4.102}$$

where C is the capacitance in the absence of a polarized body, other quantities being the same as given before.

The validity of (4.102) directly follows from (4.101). Indeed, if the voltage reduces to zero on account of the removal of charge ΔQ from the plate *1*, then the presence of this additional charge causes a voltage $\Delta U = \Delta Q/C$. The voltage ΔU is assumed to depend on changes in the potential difference between the same plates and in the same succession as for U'.

Example 4.4. A strip of thickness $b = 0.40$ cm is placed between the plates of a parallel-plate capacitor. The spacing between the plates is $h = 0.41$ cm. The capacitor was kept at a voltage $U(0-) = \varphi_1 - \varphi_2 = 4.0$ V and then disconnected from the source. The mechanical compression of the strip gave rise to the normal component of its polarization (normal to the plates from *2* to *1*) by $\Delta P = 1.55 \times 10^{-13}$ C/cm^2. What is the voltage across the capacitor after the change of polarization?

Solution. By theorem (4.102), the voltage increment is

$$\Delta U = -(\int \mathbf{E}'\Delta\mathbf{P}dV)/U'C = (E'/U'C)\Delta PV = 0.7 \text{ V}$$

The increment $\Delta U > 0$ because the vectors $\mathbf{E}$ and $\Delta\mathbf{P}$ are antiparallel.

The resultant voltage is

$$U = U(0-) + \Delta U = 4.7 \text{ V}$$

The theorem of an induced charge under the conditions of sinusoidally varying polarization. Assume the polarization varies in a simple harmonic manner and use quantities in complex notation. Resorting to formula (4.100), we immediately obtain the expression for current in a short-circuited external path (Fig. 4.36*a*):

$$\dot{I} = j\omega\dot{Q} = -(\int \mathbf{E}'j\omega\dot{\mathbf{P}}dV)/U' \tag{4.103}$$

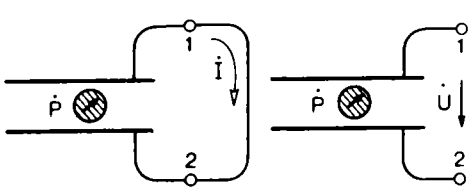


Fig. 4.36 (a) (b)

Similarly, by use of (4.102) we find the ac voltage on open-circuited plates (Fig. 4.36*b*):

$$\dot{U} = -(\int \dot{\mathbf{P}} dV)/U'C \tag{4.104}$$

The positive directions are chosen in the same manner as before.

Generalization of the theorem for a capacitor under any load conditions. From the viewpoint of the general theory of linear circuits, Eqs. (4.103) and (4.104) can be regarded as the expressions for short-circuit current $\dot{I} = \dot{I}_{sh}$ and open-circuit voltage $\dot{U} = \dot{U}_0$ of an active two-terminal network whose emf arises from the periodically varying polarization of a dielectric (Fig. 4.37*a*, *b*).

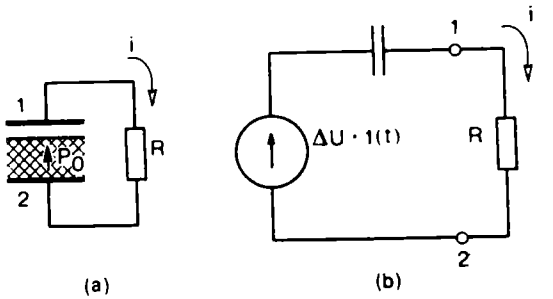


Fig. 4.37

It is known from the general theory of linear circuits that any active one-port has an internal impedance in the series equivalent circuit:

$$Z_i = \dot{U}_0/\dot{I}_{sh} \tag{4.105}$$

Consequently, in our case

$$Z_i = 1/j\omega C_0$$

This general result permits us to consider both steady periodically-varying states and transient processes.

Example 4.5[1]. The plates of the capacitor given in Example 4.4 are connected through a resistance $R = 10^8\,\Omega$ (Fig. 4.37*a*). The capacitor is not charged initially. Find how the voltage induced by $\Delta\mathbf{P}$, such as in Example 4.4, changes with time. For the solution of the problem set forth here, information is lacking in Example 4.4. Now we need to know, along with other data, the capacitance of the capacitor without a polarizable body or the equivalent plate area which is $S = 4.0\,\text{cm}^2$.

Solution. Based on what was said above, we construct an equivalent circuit (Fig. 4.37*b*) where the generator voltage $u(t) = \Delta U \cdot 1(t)$. The value of ΔU found from (4.102) is known to be equal to 0.7 V. The sought-for voltage is

$$u_R = 0.7\, e^{-\alpha t} \text{ at } \alpha = 1/RC = 1.16 \times 10^4\ \text{s}^{-1}$$

This expression applies to a transient at the generator voltage

---

[1] The solution of the same problem without resorting to the induced charge theorem is given in Sec. 3.2 [see Fig. 3.11 and Eqs. (3.71) through (3.75)].

$$\Delta u \cdot 1(t) = \begin{cases} 0 \text{ at } t \leqslant 0 \\ 0.7 \text{ V at } t > 0 \end{cases}$$

It is expected that the reader is familiar with the theory of transients in electric circuits and with the definition of the Heaviside unit:

$$1(t) = 0 \text{ at } t \leqslant 0 \text{ and } 1 \text{ at } t > 0$$

Proof of the induced charge theorem. Visualize an elemental volume of a polarized dielectric, $\mathbf{P}dV$, placed between the plates of a capacitor (Fig. 4.38). The volume element is in the shape of a pill box with a generating line $d\mathbf{l}$ parallel to the polarization vector, $d\mathbf{l} \mid \mathbf{P}$, and with sides dS normal to dl and P such that $d\mathbf{S} \mid d\mathbf{l}$. It is obvious that

$$d\mathbf{l}d\mathbf{S} = dldS = dV$$

The electric moment of the element can be expressed in terms of the two equivalent charges $\pm q$ separated from each other by the distance $d\mathbf{l}$. This moment is

$$d\mathbf{p} = \mathbf{P}dV = qd\mathbf{l} \tag{a}$$

on condition that

$$q = \mathbf{P}d\mathbf{S}$$

The system of Fig. 4.38 can be regarded as a system of four charges: two charges of a dipole, $q_1 = q$ and $q_2 = -q_1$, and two charges of the plates, q_3 and $q_4 = -q_3$. Denote the potential at a charge q_n as φ_n and assume the potential of the fourth

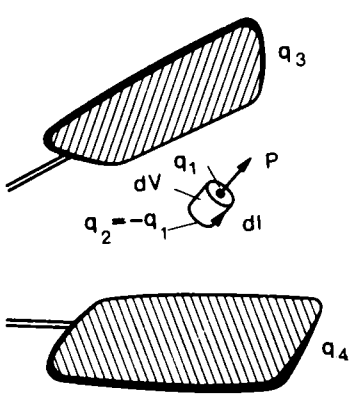


Fig. 4.38

charge equal to zero, $\varphi_4 = 0$. The relation between φ_i and q_k may then be expressed by the known system of equations (4.22) with the coefficients of potential:

$$\varphi_1 = q_1\alpha_{11} + q_2\alpha_{12} + q_3\alpha_{13}$$
$$\varphi_2 = q_1\alpha_{21} + q_2\alpha_{22} + q_3\alpha_{23}$$
$$\varphi_3 = q_1\alpha_{31} + q_2\alpha_{32} + q_3\alpha_{33}$$

where the factors α_{ik} are the coefficients of potential, obeying the reciprocity principle, $\alpha_{ik} = \alpha_{ki}$.

Bearing in mind that in our case $q_1 = -q_2 = q$, $\varphi_3 = U$ and, besides, considering the reciprocity principle just formulated above, we can write based on the given system:

$$\begin{aligned} \varphi_1 - \varphi_2 &= q(\alpha_{11} - 2\alpha_{12} + \alpha_{22}) + q_3 A \\ U &= qA + q_3\alpha_{33} \end{aligned} \qquad \text{(b)}$$

where $A = \alpha_{31} - \alpha_{32} = \alpha_{13} - \alpha_{23}$.

In the absence of a polarized element, i.e. at $q = 0$, but in the presence of the charge $q_3 = q'$ on the capacitor, the potential difference between the pill box sides is given by

$$\varphi_1' - \varphi_2' = \mathbf{E}'d\mathbf{l} = q'A$$

where $\mathbf{E}'$ is the electric field strength inside the pill box at the charge q' on the capacitor and in the absence of a polarized element, $\mathbf{P} = 0$.

From the latter equality it follows that

$$A = (\mathbf{E}'/q')d\mathbf{l}$$

If $q_3 = q_4 = 0$, i.e. the capacitor is uncharged, but $q \neq 0$, then according to (b)

$$U = qA = (\mathbf{E}'/q')qd\mathbf{l}$$

or, taking into account (a),

$$U = (\mathbf{E}'/q')\mathbf{P}dV$$

In essence, it would be more appropriate to write here dU rather than U, leaving the letter symbol U for the result obtained after integration of the latter equation through the entire volume V for which $\mathbf{E}'\mathbf{P} \neq 0$. This integral identically coincides with expression (4.102). This completes the proof of (4.102), which is equivalent to (4.100) and to other expressions that follow from (4.100) and (4.102).

Shockley's theorem. Expressions (4.100) through (4.105) that define the induced charge theorem directly lead to Shockley's theorem of the current induced by a charge q moving with a speed $\mathbf{v}$ between capacitor plates.

It should be kept in mind here that if a unit charge q actually moves with a speed $\mathbf{v}$, its motion can be expressed as a change of the moment of a dipole, $\mathbf{p} = q\mathbf{l}$, in which the other charge $-q$ remains immobile; its position (coordinates) plays no role. So,

$$\partial\mathbf{p}/\partial t = q\partial\mathbf{l}/\partial t = q\mathbf{v} \qquad (4.106)$$

Differentiating (4.100) with respect to time gives the current passing through the wire shorting out the capacitor plates:

$$i = \partial Q/\partial t = -[\int \mathbf{E}'(\partial\mathbf{P}/t)dV]/U \qquad (4.107)$$

But if within the range of action of the $\mathbf{E}$ field the only change in the dipole moment stems from the charge motion, then we have to pass from (4.107) to an expression

$$i = -(\mathbf{E}'/U')q\mathbf{v} \qquad (4.108)$$

This is one of the simplest statements of Shockley's theorem: the current in a wire short-circuiting the plates is equal to the speed of charge times the magnitude of charge and the vector of field strength $\mathbf{E}'/U'$, which would appear at the point of

location of the charge q as a result of the unit voltage across the capacitor plates (this is exactly the ratio of the field strength to the potential difference).

Reasoning in the same way as done in deriving expressions (4.103) through (4.105), we can apply Shockley's theorem to any resistance (impedance) of the circuit.

Chapter 5 Potential Electric Fields

5.1. General Features of Calculation Methods

We can always define the electric field in a homogeneous medium, given the charge distribution, by calculating the potential with formulas (1.21) that follow from Coulomb's law:

$$\varphi = [\Sigma q/R + \int (\rho dV/R) + \int (\xi/R)dS + \int (\tau/R)dl]/4\pi\varepsilon_0$$

But it is impossible to apply this method where the statement of the problem, commonly met with in electrical engineering practice, involves the given potentials at electrodes, their geometric location (Fig. 5.1*a*), and permittivity of dielectric bodies disposed between the electrodes (Fig. 5.1*b*). In such cases, recourse should be made to the field equations in differential form.

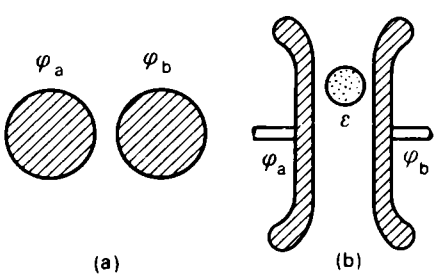


Fig. 5.1

The field calculation method based on Coulomb's law (Maxwell called it the direct method) presupposes an action at a distance: the charge $dq = \rho dV$ causes a potential component $d\varphi = dq/(\varepsilon_0 4\pi R)$ at the observation point spaced a distance R from the source point i.e. the point of location of the charge dq. Not so with the calculation based on partial differential equations. For a potential field (at $\varepsilon = 1$),

$$\nabla^2\varphi = -\rho/\varepsilon_0 \tag{5.1}$$

or, in cartesian coordinates,

$$\partial^2\varphi/\partial x^2 + \partial^2\varphi/\partial y^2 + \partial^2\varphi/\partial z^2 = -\rho/\varepsilon_0 \tag{5.1a}$$

In Sec. 95a of his treatise, Maxwell gives such a description of this equation. "In the differential equation we express that the values of the second derivatives of $V(\varphi$

in our notation) in the neighbourhood of any point, and the density at that point are related to each other in a certain manner, and no relation is expressed between the value of V at that point and the value of ρ at any point at a sensible distance from it". Thus "the differential equation is the appropriate expression for a theory of action exerted between contiguous parts of a medium". From the material presented in the preceding chapters, it is clear that now we would rather speak just of contiguous parts of a field.

Maxwell thus expressly defines the features of calculation methods employing the field laws in the form of partial differential equations and stresses their importance in the end of Sec. 95a: using these equations we shall "prepare our minds to pass from the theory of direct action at a distance to that of action between contiguous parts of a medium".

From the physical viewpoint, boundary conditions (required to solve a system of field equations) essentially influence the result only by way of exerting an effect on the field of neighbor points, but then this effect proceeds from point to point (and at a finite rate! The last circumstance is of no significance for static and quasistatic fields).

What we have said here about potential electric fields also refers to potential magnetic fields.

Integration (solution) of partial differential equations. Field calculations generally present a complicated problem. It is mostly impossible to express the calculation results in the form of simple functions. Therefore, the present book can only consider a small number of simple examples. The aim of the examples is to reflect, where possible, the most important features of the general methods of integration of field equations, paying particular attention to the physical and electrical engineering interpretation of the results. In the calculation examples discussed earlier we made use of appropriate procedures that enabled us to avoid the direct integration of differential equations.

Solution to Laplace's and Poisson's equations. The problem of calculation of irrotational fields, in which the field strength can be represented as the potential gradient, involves the solution of Laplace's and Poisson's equations. The solutions to ordinary differential equations used in, say, circuit theory become unique only after arbitrary constants (constants of integration) have been determined from the initial conditions. As for partial differential equations, the solutions include arbitrary functions as well as arbitrary constants. The choice of the functions and constants which actually correspond to the sought-for solution is done from the additional conditions that make the statement of the problem physically definite. Thus, in solving a problem for a potential electric field within a certain region, it is insufficient to say that the field in this region must satisfy Laplace's equation. For the solution to the problem we must prescribe, for example, the type of electrodes to be used to produce the field and the potentials at the electrodes (conditions at the boundary).

In the general case, to define the field in the region bounded by the surface of electrodes, it is sufficient to specify the potential at the electrodes (Dirichlet's

problem)

$$\varphi|_S = f_1(S) \tag{5.2}$$

or the normal component of the gradient (Neumann's problem)

$$\partial\varphi/\partial n|_S = f_2(S) \tag{5.3}$$

if Laplace's equation holds for the entire region and the medium that fills up this region is homogeneous.

The theorem of uniqueness for Dirichlet's problem. Assume that two solutions for φ_1 and φ_2 exist and these solutions satisfy both Laplace's equation and the given boundary conditions. Let us set up an artificial function

$$\mathbf{F} = (\varphi_1 - \varphi_2) \cdot \nabla(\varphi_1 - \varphi_2)$$

By the rule of product differentiation, its divergence is

$$\nabla\mathbf{F} = (\varphi_1 - \varphi_2) \cdot \nabla^2(\varphi_1 - \varphi_2) + [\nabla(\varphi_1 - \varphi_2)]^2$$

But the first summand is equal to zero since both φ_1 and φ_2 satisfy Laplace's equation. Hence,

$$\nabla\mathbf{F} = [\nabla(\varphi_1 - \varphi_2)]^2$$

Applying the Ostrogradsky-Gauss theorem to the found divergence, we obtain

$$\int \nabla\mathbf{F} dV = \int [\nabla(\varphi_1 - \varphi_2)]^2 dV = \oint \mathbf{F} d\mathbf{S} = \oint (\varphi_1 - \varphi_2) \cdot \nabla(\varphi_1 - \varphi_2) d\mathbf{S}$$

Extending the integral throughout all the field volume we find that the electrodes serve as a limiting surface. But according to the condition, both solutions for φ_1 and φ_2 on the electrodes take the specified value and hence are equal. Therefore,

$$\int [\nabla(\varphi_1 - \varphi_2)]^2 dV = 0$$

The integrand is everywhere positive (squared); its integral can be equal to zero only if the integrable function is zero over the total region of integration. This requirement is equivalent to the condition

$$\varphi_1 - \varphi_2 = \text{constant}$$

The constant is defined from the conditions on the electrodes at which $\varphi_1 = \varphi_2$, and hence

$$\varphi_1 - \varphi_2 = 0$$

This completes the proof of uniqueness of the solution.

Various kinds of conditions determining the unique solution to a problem. The specified conditions often differ from the ones required in solving Dirichlet's or Neumann's problems. Boundary conditions are sometimes given in a composite form. In certain problems the boundary conditions are found to be specified in terms of a tangential component of the gradient over the surface enclosing the area in question (see Sec. 7.3):

$$\partial\varphi/\partial\tau|_S = f_2(S) \tag{5.4}$$

If a field can extend into an infinite-dimensional space, the conditions at infinity are necessary along with other conditions. If we are to know the behaviour of an electric field across an interface between two different media, the specified conditions must also include boundary conditions at the interface. In the general case the problem reduces to seeking individual solutions for each medium which, in addition, must satisfy the boundary conditions. A field can be specified by the distribution of an electric polarization **P** or magnetic polarization **M** (see Secs. 5.3 and 6.1).

We will not dwell any longer on the requirements for the formulation of the conditions (the necessity and sufficiency of conditions). Let us only note that the completeness of conditions corresponds to the complete physical definiteness in the statement of the problem being solved. It is therefore obvious that in the calculation of, say, static and steady-state fields, no initial conditions are needed other than boundary conditions. On the contrary, in considering time-varying fields, even potential fields in inhomogeneous imperfect dielectrics, there is the need to specify the initial conditions too, i.e. the distribution of a charge (or potential) in the region being examined in the initial period of time, since the charge distribution changes with time.

Separate sections of this chapter deal with the method of images (Sec. 5.7) widely employed in electrical engineering and the method of complex functions (Sec. 5.8).

Other methods of field calculation. The potential field calculation methods based directly on partial differential equations are far from always being the only methods employed in practice. Of much importance is the use of integral equations and also some mathematical methods. As mentioned earlier in few instances, recent years have seen widespread application of field modeling and also simulation on analog and digital computers. The problem that often faces engineers in this connection is to reduce the initial equations to the form convenient for calculations and program preparation. The study of the methods of field calculations relates either to special technical courses or to particular branches of computing mathematics. It is hardly justifiable to include them into the general course in the theoretical bases of electrical engineering which deals only with the fundamentals of field theory.

5.2. Solution to Partial Differential Equations by the Fourier-Euler Method

The method consists in the representation of an unknown function of, let us say, two coordinates, $\varphi(x, y)$, in terms of the product or the sum of the products of functions each of which depends only on one coordinate, for example:

$$X_n = X_n(x) \text{ and } Y_n = Y_n(y) \tag{5.5}$$

The assumption here is that for any n, the expression

$$\varphi_n = X_n Y_n \tag{5.6}$$

is the solution of the equation even if it does not satisfy the given conditions which must be met only by the sum of all solutions, each being multiplied by one coefficient or another:

$$\varphi = \Sigma_n c_n X_n Y_n \tag{5.7}$$

That the sum of the solutions satisfies Laplace's equation, if each of the terms satisfies it too, becomes obvious from the linearity of the original equation $\nabla^2\varphi = 0$. Expression (5.7) obtained above is just the solution sought for since it satisfies the given boundary conditions for φ and the field equation.

By this method we obtain the desired results without running into much difficulty if it is possible to choose the coordinate system in which we can readily separate variables and set up boundary conditions. The problem then reduces to the solution of ordinary differential equations. The difficulties are most often met in seeking the function or the series of functions that satisfy the boundary conditions. In a number of cases it is even difficult to write these conditions analytically.

The method set forth here can be extended to cover the functions of three coordinates (y, x, z) or even four coordinates (y, x, z, t); in defining the dependence on t, the initial conditions must also be given. As the number of independent variables increases, it certainly becomes more and more difficult to solve the equations.

5.3. Simple Problems Solvable in the Cylindrical and Spherical Coordinate Systems

Cylinder in a uniform external field[1]. Consider a long dielectric cylinder immersed in a uniform external field E_0 so that its axis is normal to the field vector (Fig. 5.2). The cylinder has k times the permittivity of the surrounding medium. Find the resultant field in the middle portion where we can disregard the effect of the cylinder edges far distant from this portion in the direction of semiaxes $\pm z$. This is equivalent to the statement that the cylinder is infinitely long. Therefore, in the cylindrical coordinate system where the axis coincides with the cylinder axis the field proves plane-parallel, i.e. dependent only on the coordinates α and r, and identical for any value of z.

Potential distribution. Using the cylindrical coordinate system and counting off the angle α from the external field vector $\mathbf{E}_0$, we can readily prove the following: (1)

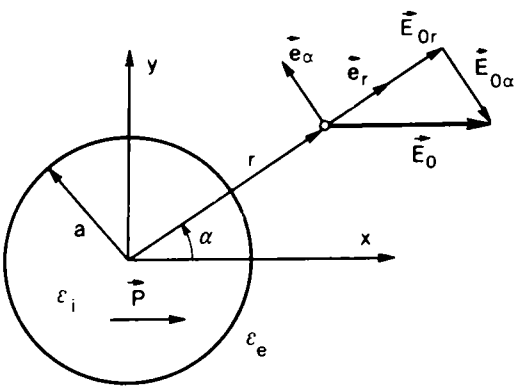


Fig. 5.2

[1] See also the text dealing with the field in the gap between the stator and rotor in Sec. 6.4 and Example 7.4.

the potential represents a periodic function of α and hence can be expressed in terms of a Fourier series; (2) coefficients of the series depend on the radius; and (3) the series contains only even functions because at any r = constant

$$\varphi(\alpha) = \varphi(-\alpha) \tag{5.8}$$

which follows from field symmetry.

Given these conditions, the Fourier series contains only cosine functions

$$\varphi(\alpha, r) = C_0 + \sum_n C_n(r) \cdot \cos n\alpha \tag{5.9}$$

The constant C_0 vanishes if φ is arbitrarily taken equal to zero in the plane $\alpha = \pm \pi/2$. We then complete the solution, being guided by the text presented in the beginning of this chapter. First we write the solution of the type given by (5.9) for the internal region, i, and external region, e:

$$\begin{aligned} \varphi_i &= \sum c_n \cos n\alpha \\ \varphi_e &= \sum C_n \cos n\alpha \end{aligned} \tag{5.10}$$

As seen from the notation itself, these solutions include different coefficients of the series, $c_1, \ldots, c_n$ and $C_1, \ldots, C_n$ which are certainly dependent on the radius.

A second step in the analysis calls for the substitution of the probable solution (5.10) into the field equation (Laplace's equation) to enable us to determine a possible form of the still unknown functions of the radius. Writing Laplace's equation in cylindrical coordinates

$$(1/r)(\partial/\partial r)[r(\partial\varphi/\partial r)] + \partial^2\varphi/r^2\ \partial\alpha^2 = 0 \tag{5.11}$$

(assuming that the potential is independent of z, see Appendix A5) and differentiating, we obtain for the external region:

$$[\partial^2 C_n(r)/\partial r^2 + \partial C_n(r)/r\partial r - (n^2/r^2)C_n(r)]\cos n\alpha = 0 \tag{5.12}$$

Since this equation must be satisfied at any value of α, the factor enclosed in the brackets should be taken equal to zero. This factor represents a differential equation containing the function of r only, differentiable with respect to r only. This analysis allows us to write the bracketed factor as an ordinary uniform differential equation

$$d^2C_n/dr^2 + (1/r)(dC_n/dr) - (n/r)^2 C_n = 0 \tag{5.13}$$

The solution to this equation is the radius to the mth power multiplied by the constant K_n,

$$C_n(r) = K_n r^m \text{ at } m = \pm n \tag{5.14}$$

which is easy to confirm by substituting (5.14) into (5.13). Really, the equation

$$[m(m-1) + m - n^2]K_n r^{m-2} = 0 \tag{5.15}$$

is identically satisfied at $m = \pm n$.

The solution for the internal region is sought in exactly the same way. The result differs from (5.14) only by constant coefficients. Let us denote them by K_n^+ and K_n^- for the external region ($r \geqslant a$) at $m = \pm n$, and $m = -n$ respectively. For the internal region ($r \geqslant a$), we denote the constants by the same lower-case letters k_n^+ and k_n^-.

So, for the internal region,

$$\varphi_i = \sum_n (k_n^+ r^n + k_n^- r^{-n}) \cos n\alpha \tag{5.16}$$

and for the external region,

$$\varphi_e = \sum_n (K_n^+ r^n + K_n^- r^{-n}) \cos n\alpha \tag{5.17}$$

Over the entire internal region, the potential must remain finite (at a finite E_0), and hence all the factors with the negative powers of r (since r may take a zero value) must be equal to zero:

$$k_n^- = 0 \tag{5.18}$$

In the external region away from the cylinder, i.e. at $r/a \gg 1$, a disturbing effect of the cylinder must vanish and the potential must be expressed as the potential of the external homogeneous (undisturbed) field. At $r \gg a$, this potential becomes then

$$\varphi_e = -E_0 x = -E_0 r \cos\alpha \tag{5.19}$$

if $\varphi = 0$ at $x = 0$, which follows from the condition $C_0 = 0$ specified earlier in formula (5.9).

The comparison between (5.19) and (5.17) leads us to conclude that

$$\begin{aligned} K_n^+ &= 0 (n \geqslant 2) \\ K_1^+ &= -E_0 \end{aligned} \tag{5.20}$$

The condition (5.19) tells us nothing of the factors when the powers of r are negative because at $r \gg a$ the corresponding terms are vanishingly small as against the first term $K_1^+ r$. The conclusions made from the comparison of (5.17) and (5.19) are just the result we have arrived at by use of the "conditions at infinity".

It still remains to use the conditions at the interface of two media. They can be expressed as a potential equality

$$\varphi_e = \varphi_i \text{ at } r = a \tag{5.21}$$

and as the equality of normal components of the electric flux density vector. Since in our case the normal components are radial ones, the latter condition can be stated as

$$\varepsilon_i \partial\varphi_i/\partial r = \varepsilon_e \partial\varphi_e/\partial r \text{ at } r = a \tag{5.22}$$

Taking into consideration (5.18) and (5.20), we rewrite the potential expressions for the sake of clarity:

$$\varphi_i = \sum_{n \geqslant 1} k_n^+ r^n \cos n\alpha \tag{5.23}$$

$$\varphi_e = -E_0 r \cos\alpha + \sum_{n \geqslant 1} K_n^- r^{-n} \cos n\alpha \tag{5.24}$$

Proceeding from (5.21) and (5.22), we readily pass to the following systems of equations:

at $n = 1$,

$$\begin{aligned} k_1^+ a &= -E_0 a + K_1^-/a \\ \varepsilon_i k_1^+ &= -\varepsilon_e (E_0 + K_1^- a^{-2}) \end{aligned} \tag{5.25}$$

and at $n \geqslant 2$,

$$k_n^+ a^n = K_n^-/a^n$$
$$\varepsilon_i n k_n^+ a^{n-1} = -\varepsilon_e n K_n^- a^{-(n+1)} \tag{5.26}$$

From two Eqs. (5.25) in two unknowns we find

$$k_1^+ = -E_0 2\varepsilon_e/(\varepsilon_i + \varepsilon_e)$$
$$K_1^- = E_0 \lambda a^2 \tag{5.27}$$

where $\lambda = (\varepsilon_i - \varepsilon_e)/(\varepsilon_i + \varepsilon_e)$.

Equations (5.26) lead us to conclude that

$$K_n^- = k_n^+ = 0 \text{ at } n \geqslant 2$$

Having found all the coefficients, we come to the final expressions for potentials:

in the external region ($r \geqslant a, \varepsilon = \varepsilon_e$),

$$\varphi_e = -E_0[1 - \lambda a^2/r^2] r \cos\alpha \tag{5.28}$$

in the internal region ($r \leqslant a, \varepsilon = \varepsilon_e$),

$$\varphi_i = -E_0 2\varepsilon_e/(\varepsilon_i + \varepsilon_e)] r \cos\alpha \tag{5.29}$$

The expressions for the field strength components become:

in the external region,

$$E_{e,r} = -\partial\varphi_e/\partial r = E_0[1 + \lambda a^2/r^2]\cos\alpha \tag{5.30a}$$
$$E_{e,\alpha} = -\partial\varphi_e/r\partial\alpha = -E_0[1 - \lambda a^2/r^2]\sin\alpha \tag{5.30b}$$

in the internal region,

$$E_i = E_{i,x} = -\partial\varphi_i/\partial x = E_0[2\varepsilon_e/(\varepsilon_i + \varepsilon_e)] \tag{5.30c}$$

In Eq. 5.30 *c*, x stands for the cartesian coordinate $x = r\cos\alpha$ whose direction coincides with that of E_0. This is the only component of the field which remains independent of coordinates inside the cylinder (E = constant at $r \leqslant a$).

The found result explains why the field strength can noticeably rise inside inclusions with ε lower than that of the surrounding medium. This sometimes fosters breakdown or insulation damage due to a discharge originating inside an inclusion. For example, the field strength inside a threadlike air bubble ($\varepsilon_i = 1$) in a piece of glass ($\varepsilon_e = 7$) exposed to a transverse external field can be $2\varepsilon_e/(\varepsilon_e + \varepsilon_i) = 1.75$ times the field strength that would be found in pure glass. In the limit, if $\varepsilon_e \gg \varepsilon_i$, the field strength inside an inclusion becomes twice as high. At $\varepsilon_e \ll \varepsilon_i$, the field strength increases twofold (in comparison with the undisturbed field strength) on the outside of the cylinder at its "poles" (at points $r = a$; $\alpha = 0$ or π).

The found solution is easy to apply to problems relating to a conducting cylinder in a uniform external field. For this it is enough to set formally $\varepsilon_i \to \infty$ in (5.30a) and (5.30b), for the field inside the cylinder vanishes under these conditions, as is the case for a conductor. Outside the conductor ($r \geqslant a$), λ is equal to unity, and hence

$$E_{e,r} = E_0[1 + a^2/r^2]\cos\alpha$$
$$E_{e,\alpha} = -E_0[1 - a^2/r^2]\sin\alpha \tag{5.30d}$$

Sphere in a uniform external field. As with the cylinder, the sphere is taken to have a permittivity ε_i and the surrounding medium a permittivity ε_e. It is advantageous to choose the spherical coordinate system with the z axis running along the vector of the external field E_0 (Fig. 5.3).

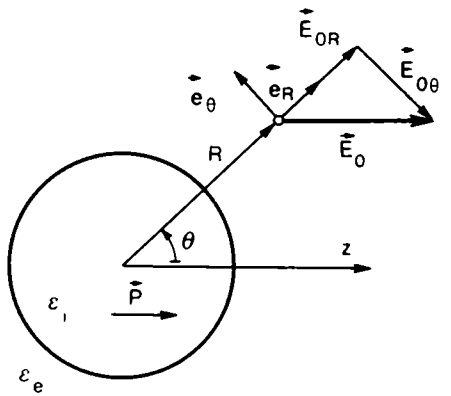


Fig. 5.3

The field depends on two coordinates, radius R and azimuth angle θ, and does not depend on the coordinate α (meridian angle). This means that the field is the same in any meridian plane (α = constant). Such a field is called plane-meridian.

Proceeding from the earlier considerations of the periodic dependence of the angle, we obtain the general solution in such a form

$$\varphi = (AR + BR^{-2})\cos\theta \tag{5.31}$$

if the assumption is that the potential is equal to zero in the equatorial plane ($\varphi = 0$ at $\theta = \pi/2$). Here, A and B are constants.

It is easy to prove the above statement by substituting φ into Laplace's equation expressed in spherical coordinates (see Appendix A5).

The constants A and B are found from the same conditions as those specified for the preceding case (the cylinder in a uniform field). We can now specify the dielectric sphere of radius a immersed in the external homogeneous field E_0. The potential in a region outside the sphere is

$$\varphi_e = -E_0[1 - \eta a^3/R^3]R\cos\theta \tag{5.32}$$

where $\eta = (\varepsilon_i - \varepsilon_e)/(\varepsilon_i + 2\varepsilon_e)$.

The potential inside the sphere is given by

$$\varphi_i = -E_0[3\varepsilon_e/(\varepsilon_i + 2\varepsilon_e)]R\cos\theta \tag{5.33}$$

The field strength components are:

in the external region,

$$E_{e,R} = -\partial\varphi_e/\partial R = E_0[1 + 2\eta a^3/R^3]\cos\theta \tag{5.34a}$$

$$E_{e,\theta} = -\partial\varphi_e/R\partial\theta = -E_0[1 - \eta a^3/R^3]\sin\theta \tag{5.34b}$$

in the internal region,

$$E_i = E_z = -\partial\varphi/\partial z = E_0[3\varepsilon_e/(\varepsilon_i + 2\varepsilon_e)] \tag{5.34c}$$

where $z = R\cos\theta$.

As with the cylinder discussed earlier, the field inside and outside the sphere can differ substantially from the external uniform field. Thus at $\varepsilon_e > \varepsilon_i$, the field inside the sphere is larger than E_0; if $\varepsilon_e \gg \varepsilon_i$, then $E_i \approx 1.5E_0$. If $\varepsilon_e < \varepsilon_i$, the field inside the sphere weakens, but rises outside, particularly at the sphere poles ($\theta = 0$ or π).

To determine how a conducting sphere immersed in a uniform field distorts the field pattern, the sphere permittivity ε_i may be set at infinity:

$$E_{e,R} = E_0[1 + 2a^3/R^3]\cos\theta$$
$$E_{e,\theta} = -E_0[1 - a^3/R^3]\sin\theta \tag{5.34d}$$

Note that whether we consider a cylinder or a sphere, at $\varepsilon_i = \varepsilon_e$ all the expressions take the form of the expression for an undisturbed field. Taking all the permittivities related to ε_e, we can simplify somewhat the obtained expressions and write:

for a cylinder,

$$\varphi_i = -E_0[2/(\varepsilon + 1)]r\cos\alpha \quad \text{and} \quad E_i = E_0[2/(\varepsilon + 1)]$$

for a sphere,

$$\varphi_i = -E_0[3/(\varepsilon + 2)]R\cos\theta \text{ and } E_i = E_0[3/(\varepsilon + 2)]$$

where $\varepsilon = \varepsilon_i/\varepsilon_e$; η and λ remain the same.

Graphical construction of equipotentials by known equations. It might be a very useful exercise for the student to try to construct field equipotentials for any of the cases discussed in this chapter.

Since the potential is defined by two coordinates,

$$\varphi = \varphi(\xi, \rho) \tag{5.35}$$

the construction should be done in the following manner. Setting the potentials such as $\varphi_1, \varphi_2 \ldots$ and any concrete values of one of the coordinates, for example, ξ_1, $\xi_2 \ldots$, we find the coordinates, $\rho_1, \rho_2 \ldots$ i.e. locate the points at which the given equipotentials $\varphi_1, \varphi_2 \ldots$ cross the lines $\xi_1, \xi_2 \ldots$ It is generally advisable to consruct equipotentials at a definite potential difference between neighbor equipotential lines.

Comparison of the results obtained for the cylinder and sphere. It is apparent from solutions (5.30d) and (5.34d) that (a) if a conductor such as a cylinder or a sphere is held in a uniform external field E_0, the field strength at the poles ($r = a, \alpha = 0; \pi$ or $R = a; \theta = 0$ or π) of the cylinder and the sphere rises to $2E_0$ and to $3E_0$ respectively; and (b) if a conducting medium has a hole (or $\varepsilon_i \ll \varepsilon_e$), the field strength inside a cylindrical hole [see Eq. (5.30c)] grows to $2E_0$ and inside a spherical hole [see Eq. (5.34c)] to merely $1.5\,E_0$, where E_0 is the external uniform field strength.

A greater effect of the conducting sphere in one case (a) and a greater effect of the nonconducting cylinder in the other (b) can readily be accounted for by applying the general concepts of the flux of electric field lines. In the case (a), the flux lines strive to find an easy way to pass through the conducting inclusion, so that the lines converge toward the sphere, heading from all the sides for its pole. In the case (b),

however, the lines strive for the cylinder surface and do not cross anywhere the planes normal to the cylinder axis. It is thus natural to expect that the flux lines will be denser near the sphere poles ($3E_0$) than in the cylinder ($2E_0$). In the case (b), on the contrary, the flux lines must bypass the obstacles on their way. Obviously, it is easier to go round the sphere (practically on all the sides) than to move round the transversely lying cylinder. That is why the density of flux lines inside the cylinder is higher ($2E_0$) than that inside the sphere ($1.5E_0$).

One should note the following fact. As the field strength E_i inside an inclusion increases (or decreases), the electric flux density $D_i = \varepsilon_i \varepsilon_0 E_i$ inside the same inclusion decreases (or rises respectively).

Field components due to polarization. In the cases considered above for the dielectric cylinder or sphere immersed in an external uniform field E_0, a change in the field arises from an additional field of charges due to polarization. If $\varepsilon_e = 1$, the charges result only from the polarization of a dielectric in the form of a cylinder or sphere. The resultant field may be taken to equal the sum of external field $\mathbf{E}_0$ and additional field $\mathbf{E}_P$ of the polarized body:

$$\mathbf{E} = \mathbf{E}_0 + \mathbf{E}_P \tag{5.36}$$

Since we know the resultant field components and also the components of the external undisturbed field, it is always easy to determine the polarization field components:

$$\mathbf{E}_P = \mathbf{E} - \mathbf{E}_0$$

Cylinder ($\varepsilon_i = \varepsilon$, $\varepsilon_e = 1$). For the internal region ($r \leqslant a$), formula (5.30) gives

$$E_P = E_0[2/(\varepsilon + 1)] - E_0 = -E_0[(\varepsilon - 1)/(\varepsilon + 1)] \tag{5.37}$$

The field of polarization is antiparallel ($\varepsilon > 1$) to the external field.

As regards the external region, we need first to evaluate the components of the undisturbed field $\mathbf{E}_0$ in cylindrical coordinates (see Fig. 5.2):

$$\begin{aligned} E_{0,\alpha} &= -E_0 \sin\alpha \\ E_{0,r} &= E_0 \cos\alpha \end{aligned} \tag{5.38}$$

and then, using (5.30) and (5.38), to determine

$$E_{P,r} = E_0[(\varepsilon - 1)/(\varepsilon + 1)](a/r)^2 \cos\alpha \tag{5.39a}$$

$$E_{P,\alpha} = E_0[(\varepsilon - 1)/(\varepsilon + 1)](a/r)^2 \sin\alpha \tag{5.39b}$$

at $r \geqslant a$.

The field strength due to polarization is easy to express directly in terms of $\mathbf{P}$. For this we should recall that $\varepsilon - 1$ is the susceptibility k_e. The polarization of the dielectric cylinder is then

$$P = E_i \varepsilon_0 (\varepsilon - 1) \tag{5.40}$$

Using (5.30c), we find that at $\varepsilon_e = 1$ and $\varepsilon_i = \varepsilon$

$$P = E_0[2/(\varepsilon + 1)]\varepsilon_0(\varepsilon - 1) \tag{5.41}$$

This formula permits us to express E_0 in terms of P in (5.37) and (5.39), following

which we get:
for the internal region ($r \leqslant a$),

$$E_P = -P/2\varepsilon_0 \tag{5.42a}$$

for the external region ($r \geqslant a$),

$$E_{P,r} = p\cos\alpha/2\pi\varepsilon_0 r^2 = Pa^2\cos\alpha/2\varepsilon_0 r^2 \tag{5.42b}$$

$$E_{P,\alpha} = p\sin\alpha/2\pi\varepsilon_0 r^2 = Pa^2\sin\alpha/2\varepsilon_0 r^2 \tag{5.42c}$$

where $p = P \cdot \pi a^2$ is the moment per unit length.

Sphere ($\varepsilon_i = \varepsilon$, $\varepsilon_e = 1$). Repeating the calculations in the same way as for the cylinder we arrive at the expression for polarization

$$P = E_0[3/(\varepsilon + 2)]\varepsilon_0(\varepsilon - 1) \tag{5.43}$$

and the expression for the electric field in the internal region ($R \leqslant a$)

$$E_P = -P/3\varepsilon_0 \tag{5.44}$$

As seen $\mathbf{E}_P$ is constant and acts in the opposite direction to the vector $\mathbf{P}$.

Such a direction of $\mathbf{E}_P$ is obvious from the illustration of charges due to polarization on the sphere surface (Fig. 5.4). These charges and their field are related by Coulomb's law.

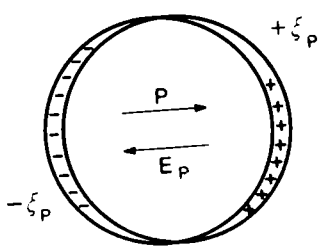


Fig. 5.4

For the external region ($R \geqslant a$), the field strength of the polarized sphere is expressed as

$$E_{P,R} = p\cos\theta/2\pi\varepsilon_0 R^3 = 2Pa^3\cos\theta/3\varepsilon_0 R^3$$
$$E_{P,\theta} = p\sin\theta/4\pi\varepsilon_0 R^3 = Pa^3\sin\theta/3\varepsilon_0 R^3 \tag{5.45}$$

where $p = PV = P \cdot 4\pi a^3/3$ is the total moment of the sphere.

It is interesting to note that these expressions and expressions (1.25) for the field of a dipole with moment p are identical.

Field of polarized bodies. The above expressions for field components due to polarization are certainly true regardless of the causes that lead to polarization P. In other words, they also hold if the polarization of a body results from the influences other than the field that acts at the given moment, i.e. $P \neq E_i\varepsilon_0(\varepsilon - 1)$. Polarization most often arises from mechanical deformation or appears as a residual effect of the external field that previously acted on a nonlinear medium. One can liken this effect to residual deformation of a bent bar when the stress goes beyond the limit of elasticity.

Instead of using (5.36) and other E field expressions derived earlier, we can apply Laplace's equations directly to determine the field from the given polarization for new boundary conditions. First, in the new problem the potential and the field at infinity are zero (if the potential in the midplane is taken equal to zero) because the external field is nonexistent and polarized bodies can cause a noticeable disturbance only for a short distance. Second, the continuity condition for the normal component of the electric flux density vector (for a cylinder) is now set up using the explicit expression of polarization:

$$D_{e,r} = -\varepsilon_0(\partial\varphi_e/\partial r) = D_{i,r} = -\varepsilon_0(\partial\varphi_i/\partial r) + P_r \tag{5.46}$$

In the problem relating to a cylinder, $P_r = P\cos\alpha$. The condition for the potential at the interface remains the same. It is very useful for the student to tackle this solution on his own.

5.4. Polarized Ellipsoid. Coefficients of Depolarization

The field of a uniformly polarized ellipsoid (spheroid) can be calculated using Laplace's equations and the boundary conditions in the elliptical coordinate system suitable for the purpose. However, the entire calculation process and the functions that describe the field outside the ellipsoid are rather complex. It is possible to obtain simple expressions only for the field inside a uniformly polarized ellipsoid or the field of an ellipsoid immersed in an external uniform field (Fig. 5.5). The field inside an ellipsoid is homogeneous due to its uniform polarization. This is the property intrinsic only in the bodies shaped as an ellipsoid.

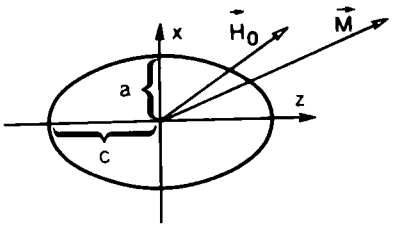


Fig. 5.5

The components due to polarization P that appear inside a spheroid are given by

$$\begin{aligned} E_x &= -N_x P_x/\varepsilon_0 \\ E_y &= -N_y P_y/\varepsilon_0 \\ E_z &= -N_z P_z/\varepsilon_0 \end{aligned} \tag{5.47}$$

The factors N_j are known as the *coefficients of depolarization.*

For a spheroid with the z axis of rotation, the semiaxes in the x and y directions are equal. From the considerations of symmetry it is obvious that the corresponding coefficients are also equal:

$$N_x = N_y \tag{5.48}$$

The resultant field inside the spheroid exposed to an external field with components E_{0x} and E_{0z} is expressed as

$$\begin{aligned} E_x &= E_{0x} - N_x P_x/\varepsilon_0 \\ E_y &= -N_y P_y/\varepsilon_0 \\ E_z &= E_{0z} - N_z P_z/\varepsilon_0 \end{aligned} \tag{5.49}$$

In an isotropic linear dielectric, $P_{x,z} = \varepsilon_0 k E_{x,z}$. The field inside the ellipsoid is then given by

$$\begin{aligned} E_x &= E_{0x}/(1 + kN_x) \\ E_y &= 0 \\ E_z &= E_{0z}/(1 + kN_z) \end{aligned} \tag{5.50}$$

For an elongated ellipsoid of rotation ($a \leqslant c$)

$$N_z = (1 - \gamma^2)\,(\operatorname{Artanh}\gamma - \gamma)/\gamma^3 \tag{5.51a}$$

$$N_x = N_y = [\gamma - (1 - \gamma^2)\operatorname{Artanh}\gamma]/2\gamma^3 = (1 - N_z)/2 \tag{5.51b}$$

where

$$\gamma = \sqrt{1 - (a/c)^2}$$

In computations, it may prove useful to keep in mind that

$$\operatorname{Artanh}\gamma = \ln[(1 + \gamma)/(1 - \gamma)]/2$$

For an oblate ellipsoid of rotation ($a \geqslant c$)

$$N_z = (1 + \beta^2)(\beta - \arctan\beta)/\beta^3 \tag{5.52a}$$

$$N_x = N_y = [(1 + \beta^2)\arctan\beta - \beta]/2\beta^3 = (1 - N_z)/2 \tag{5.52b}$$

where

$$\beta = \sqrt{(a/c)^2 - 1}$$

For ellipsoids of any shape,

$$N_x + N_y + N_z = 1 \tag{5.53}$$

The knowledge of the coefficient N_j for an ellipsoid is of practical importance mainly because real bodies of revolution commonly met in practice are not perfect ellipsoids. For example, a disk can be regarded as an oblate spheroid, and a round core as an elongated spheroid. This assumption allows us to determine the coefficients N for these bodies.

Strictly speaking, the notion of a depolarization coefficient is only applicable to ellipsoids, although it is sometimes given a wider interpretation (see Sec. 6.2).

On coefficients for a sphere, cylinder, and plate. On the strength of symmetry and Eq. (5.53), we can obviously arrive at Eqs. (5.42a) and (5.44) deduced earlier: for a sphere, $N = 1/3$, and for an infinitely long round cylinder exposed to a transverse field, $N = 1/2$. In the latter case, the major axis of the ellipsoid is $c \to \infty$. Besides, $N_z \to 0$ because the depolarizing effect of polarization charges uncovered at infinitely remote ends of the elongated ellipsoid is equal to zero. The same considerations permit us to conclude that the depolarization coefficient for an infinitely long plate in a transverse field is $N = 1$.

5.5. Polarizable Bodies in an AC Field

An ac field in polarizable media may often be thought of as a quasistatic field, i.e. the field that obeys the Laplace-Poisson equation. In considering the steady-state conditions of a sinusoidally varying field, however, we should convert to the complex presentation of field vectors and potentials and also of the parameters of a medium. In discussing transients, it is expedient to resort to appropriate operator expressions.

Dielectric sphere with complex permittivity in an external ac field at $\varepsilon_e = 1$. For the case at hand, all the formulas derived in this chapter apply equally well if all the variables such as φ, **E**, **D**, **P** are given in complex notation as is the sphere permittivity $\underline{\varepsilon}_i = \underline{\varepsilon} = \varepsilon' - j\varepsilon''$ that can also be written as

$$\underline{\varepsilon}_i = \varepsilon' - j\sigma/\varepsilon_0\omega \tag{5.54}$$

It is of interest to note a certain feature that shows up in an imperfect dielectric ($\varepsilon'' \neq 0$ or $\sigma \neq 0$). At points lying outside the sphere ($R \geqslant a$), the components of external uniform field $\dot{\mathbf{E}}_0$ and polarization field $\dot{\mathbf{E}}_P$ differ in direction and phase.

Indeed, by expression (5.45),

$$\dot{\mathbf{E}}_P = \dot{E}_0[2\cos\theta \cdot \mathbf{e}_R + \sin\theta \cdot \mathbf{e}_\theta]\underline{F} \tag{5.55}$$

where

$$\underline{F} = [(\underline{\varepsilon} - 1)/(\underline{\varepsilon} + 1)](a/R)^3 \tag{5.56}$$

whereas (see Fig. 5.3)

$$\dot{\mathbf{E}}_0 = \dot{E}_0[\cos\theta \cdot \mathbf{e}_R - \sin\theta \cdot \mathbf{e}_\theta] \tag{5.57}$$

Such a field contains rotational components.

A cylinder, ellipsoid, and other bodies in an ac field. For a cylinder and an ellipsoid, we should employ the same procedure as for the sphere discussed above, i.e. use the available solutions in complex form.

Of interest is the fact that the field inside a sphere and cylinder (exposed to an external uniform field, of course) does not show rotational components. In the case of complex parameters of a medium, however, circular components appear inside an ellipsoid too, though they are insignificant.

5.6. Transients in Quasistatic Fields. Relaxation

Operator method for calculation of the field in an inhomogeneous imperfect dielectric. The dependence of the equivalent parameters of a medium on frequency (see Ch. 3) shows that the field distribution in a nonuniform medium generally differs with the frequency of an external field. If the response depends on the frequency of a disturbing field, we have to expect a transient process to take place as a result of an aperiodic disturbance. Really, an aperiodic disturbance contains a definite spectrum, so that different frequency components cause different responses.

To describe a transient, it is convenient to resort to the operator expression for the equivalent permittivity ε_{eq} (or σ_{eq}) replacing $j\omega$ by p in accordance with (3.57) and (3.69):

$$\varepsilon_{eq} = \varepsilon_{eq}(p) = \varepsilon + \sigma/p\varepsilon_0 \tag{5.58}$$

It is certain that the external disturbance must be presented in operator notation. If the calculation results in an operator expression, say, $\mathbf{E}(p)$, for any field point, then, following the general rules of the operator method, we can find the appropriate time function

$$\mathbf{E}(t) \sim \mathbf{E}(p)$$

The calculation may involve certain difficulties if the initial conditions cannot be taken as zero conditions. For the analysis of the problem, it is first necessary to consider a simple concrete example for which, as will be shown later, the initial conditions are zero ones.

Imperfect dielectric sphere in a surrounding medium which is also an imperfect dielectric. Let us immerse a solid resin sphere of radius a having parameters ε_i and σ_i in a gaseous medium with parameters ε_e and σ_e. Assume that $\varepsilon_i = 2\varepsilon_e$, $\varepsilon_e = 1$, $\sigma_i = 0.1\sigma_e$, and $\sigma_e = 10^{-16}$ S/cm.

The external field $\mathbf{E}_0$ that acts on the system referred to is homogeneous. The dielectric sphere must intensify the field $\mathbf{E}$ near the sphere poles (because $\varepsilon_i > \varepsilon_e$) in such a manner as shown in Fig. 5.6.

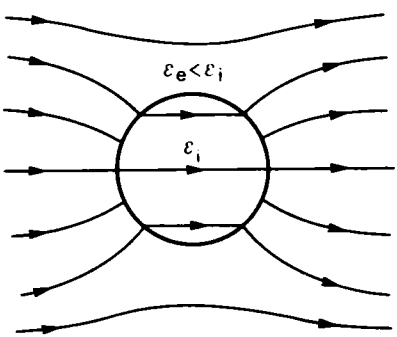


Fig. 5.6

Consider the field outside the sphere ($R \geqslant a$) just at its poles. Using (5.34) and setting $\theta = 0$ and $R = a$, we find that $E = E_R = 1.5\,E_0$ for the parameters specified above.

But we have neglected here the effect of conductivities, σ, and made the calculation entirely on the basis of the relation between the permittivities, ε, of the sphere and the environment. Is it safe to do so? Yes, it is, but only for the moments of time that are very close to the moment of application of an external field. On applying the external field, $\mathbf{E}_0$, the dielectric polarizes very fast[1] to give rise to bound charges on the sphere surface. These charges lead to a field distribution whose pattern now depends on the permittivities.

In the general case[2], along with bound charges, free charges can also appear in a nonuniform medium that may change the field pattern completely. Free charges

[1] By taking ε purely real and independent of ω, we thus assume that P follows in step with field changes.

[2] If the ratio σ/ε varies throughout the medium, i.e. if $\varepsilon_i/\sigma_i \neq \varepsilon_e/\sigma_e$.

begin to accumulate gradually and indeed very slowly in a medium of low conductivity. As the process comes to an end, the field distribution only depends on the relation between the conductivities of the sphere and surrounding medium. To describe the field so produced, ε_i and ε_e in (5.34) should be replaced by ε_{eq} of (3.57); if $t \to \infty$, ω may be taken equal to zero. On performing requisite cancellations, the expressions for E at $R \geqslant a$ take the form

$$E_{e,R} = E_0[1 + 2\eta a^3/R^3]\cos\theta \tag{5.59a}$$

$$E_{e,\theta} = -E_0[1 - \eta a^3/R^3]\sin\theta \tag{5.59b}$$

where $\eta = (\sigma_i - \sigma_e)/(\sigma_i + 2\sigma_e)$.

For the case at hand, with the field distribution being determined by the conductivities, we find that in the external region ($R \geqslant a$), $E = 0.145E_0$ (Fig. 5.7) near the pole ($R = a$, $\theta = 0$).

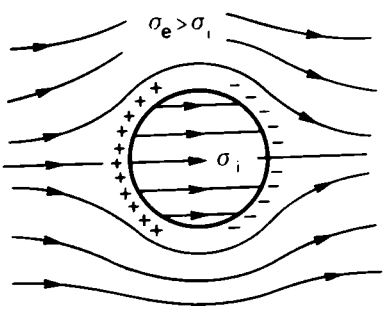


Fig. 5.7

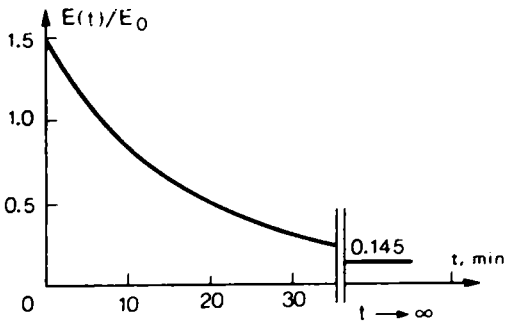


Fig. 5.8

Introduce now equivalent operator permittivities given by (5.58) into (5.34) and express the applied field in operator form $E_0(t) = E_0 \cdot 1(t)$ to obtain the operator expression for the field strength of interest under zero initial conditions. The conditions are indeed the zero ones because at the initial moment free charges do not exist in the field. To find the solution, we apply the known methods of operational calculus widely used in circuit theory. Substituting the above expressions for ε and σ into (5.58), (5.32), and (5.34a), we find that in the external region ($R \geqslant a$) just near the sphere ($R \to a$, $\theta = 0$), the field is

$$E = E_R(t) = E_0(0.145 + 1.35e^{-kt})$$

at $k = 1.18 \times 10^{-3}\text{s}^{-1}$.

The dependence $E(t)/E_0$ is shown in Fig. 5.8. The field strength drops from $1.5E_0$ (at the initial moment) to $0.8E_0$ in ten minutes; 30 min after the application of the external field, the field strength decays to $0.3E_0$.

5.7. Method of Images

This method was mainly worked out for handling certain problems in electrical engineering. It greatly extended the class of easily solvable problems.

Charge images. The method of images can be applied to determine the $+q$ located near a conducting surface with $\sigma > 0$ (Fig. 5.9). For this, we replace the effect of the entire surface (the effect of charges induced on it) by the field of an image of the given charge with the opposite sign, $-q$. Indeed, in the field of such two charges, the midplane (ground, or image plane) proves to be an equipotential surface (from the conditions of symmetry), which agrees with the requirements of actual conditions. The method of images was evolved by a great British physicist Kelvin back in the middle of the last century, and is still widely used in practice.

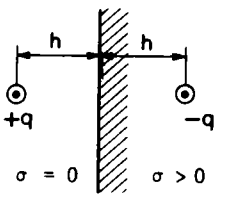


Fig. 5.9

The method owes its substantial extention to other British physicist Searle who applied it to the cases where the image plane represents the interface between any two different media.

Consider a charge q (Fig. 5.10*a*) placed in a medium of permittivity ε_1 at a distance h from a plane boundary between this medium and a medium with permittivity ε_2. The question is whether we can define the field assuming that the field in the first region is due to the actual charge q and to its incomplete (imperfect) image $q_1 = k_1 q$ lying a distance h away from the boundary. The assumption also is that both charges lie in a uniform unbounded medium of permittivity ε_1 (Fig. 5.10*b*).

Reasoning further from the analogy between real objects and mirror images, we may assume that the field in the second region is produced by a certain charge q_2 located at the point of the actual charge q but being quantitatively different from it, namely, $q_2 = k_2 q$. It may also be assumed that the entire space is filled with a homogeneous medium of permittivity ε_2 (Fig. 5.10*c*) [1]. To support the validity of the above assumptions let us start with certain reasonings from which we will then determine the coefficients k_1 and k_2.

If the field in the first region is defined as the field of charges q and q_1 at a medium permittivity ε_1 and the field in the second region is defined as the field of charge q_2 at a medium permittivity ε_2, then the two solutions for the fields at the boundary between the two regions must satisfy the following boundary conditions:

$$E_{1t} = E_{2t}, \quad \varepsilon_1 E_{1n} = \varepsilon_2 E_{2n} \tag{5.60}$$

---

[1] Searle applied the analogy between a candle (q) burning in the room and its image incompletely reflected in the window glass (q_1). Imagine fog ($\varepsilon = \varepsilon_2$) outside the window. The candle will obviously appear to be more dimly visible on the other side (q_2), as if clouded in fog (ε_2).

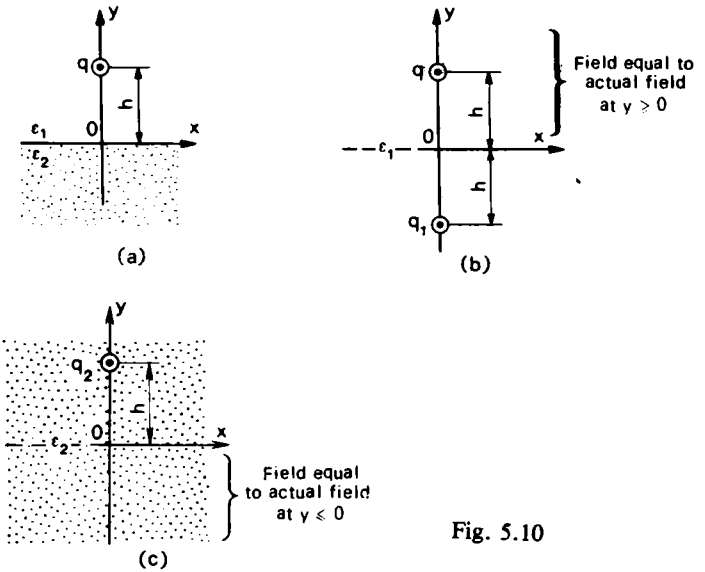


Fig. 5.10

With the assumptions made for the first region, $y \geqslant 0$ (see Fig. 5.10*b*), the potential assumes the form

$$\varphi_1 = (q/4\pi\varepsilon_0\varepsilon_1)\{[(y-h)^2 + x^2 + z^2]^{-1/2} + k_1[(y+h)^2 + x^2 + z^2]^{-1/2}\}$$

where the factors in brackets are the squares of the distances from the observation point to charges q and $q_1 = k_1 q$, if the xz plane coincides with the interface and the y axis passes through the charge q.

For the second region, $y \leqslant 0$ (see Fig. 5.10*c*),

$$\varphi_2 = (q_2/4\pi\varepsilon_0\varepsilon_2)[(y-h)^2 + x^2 + z^2]^{-1/2}$$

where $q_2 = k_2 q$.

Setting $\varphi_1 = \varphi_2$ at $y = 0$, we will find that this condition satisfies the first equation in (5.60) for the tangential **E** components. After canceling, we obtain from equalities (5.60)

$$1 + k_1 = k_2\varepsilon_1/\varepsilon_2' \tag{5.61}$$

The condition in (5.60) for normal components will be expressed as derivatives with respect to y at $y = 0$:

$$\varepsilon_1 \partial\varphi_1/\partial y = \varepsilon_2 \partial\varphi_2/\partial y$$

whence

$$1 - k_1 = k_2 \tag{5.62}$$

This result together with (5.61) leads us to state that

$$k_1 = (\varepsilon_1 - \varepsilon_2)/(\varepsilon_1 + \varepsilon_2)$$
$$k_2 = 2\varepsilon_2/(\varepsilon_1 + \varepsilon_2) \tag{5.63}$$

The image charge $q_1 = k_1 q$ takes a sign depending on the relation between ε_1 and ε_2. At $\varepsilon_1 > \varepsilon_2$, the field lines extend so as if they avoided entering the second region ($k_1 > 0$); at $\varepsilon_1 < \varepsilon_2$, on the contrary, it is as though the lines were drawn into the second region ($k_1 < 0$). Notice that for a uniform field where $\varepsilon_1 = \varepsilon_2$, we have $k_2 = 1$ and $k_1 = 0$, which is to be the case.

If the second region is a conducting medium, then the condition is formally equivalent to the condition $\varepsilon_2 \to \infty$, in which case $k_1 = -1$, i.e. the charge changes sign and its reflection is complete.

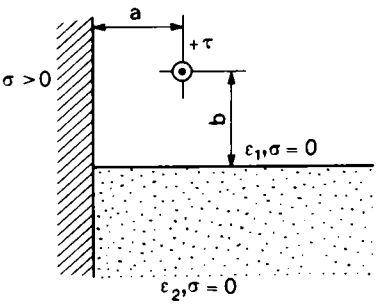


Fig. 5.11

Figure 5.11 shows the transverse cross section of a wire with a charge $+\tau$. The wire runs parallel to a conducting wall and to the surface of a dielectric, say, liquid oil of permittivity ε_2 that fills a metallic tank. The field above the dielectric surface can be specified by the method of double reflection: (1) the charge $+\tau$ is reflected in the dielectric so that its image charge is $+\tau_1$ and (2) both charges are reflected in the wall, the image charges being $-\tau$ and $-\tau_1$. The field above the dielectric surface is represented as the field of the four charged wires in the unbounded dielectric with ε_1.

Method of images for nonuniform conductors. The generalized formulas (3.56) and (3.57) for imperfect dielectrics given in Sec. 3.2 allow us to apply all the solutions found above to conducting media. Let it be required to determine the conductance per unit length between parallel reinforcement bars of Fig. 5.12*a*, which illustrates the cross section of a concrete foundation infinitely extending to the left.

Using the method of images, we can determine the sought-for field if we replace the nonconducting half of the space ($\sigma_2 = 0$) by a limitless medium of the same conductivity σ_1 as that of the first medium on the left (Fig. 5.12*b*). The medium on the right thus contains the images of the electrodes (bars) whose charges do not change sign. Such a reflection corresponds to the absence of the normal components of the current density on the surface of the first medium and hence to the absence of the normal component of field strength given by (3.129). The current distribution in the system under study is shown in Fig. 5.12*b*. Replacing in the calculation formulas for parallel wires (see Sec. 4.1) τ' by I and $\varepsilon\varepsilon_0$ by σ, yields the relation between the current I and the potential difference between the real electrodes, $U = 2\varphi$. The ratio between these quantities is equal to the conductance $G = I/2\varphi$.

Considering the forces acting on the charge q by Coulomb's law, we should take into account not only the charges and dipoles that exist independently but also the

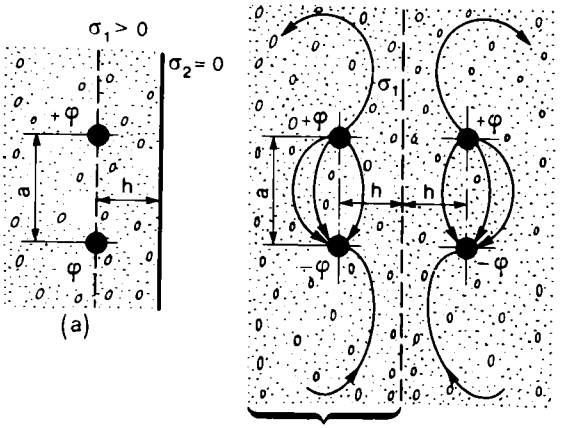


Fig. 5.12

charges originating in conductors and dielectrics under the effect of the charge q being examined. The effect of the latter charges comes to the interaction of these charges with their images. Whenever a charge strives to escape the conductor (the effect is known as electron or ion emission) the forces of image charges always tend to hold the charge.

5.8. Method of Complex Functions. Conformal Mapping

The method is applicable to plane-parallel fields in which the potential depends only on two cartesian coordinates, say, $\varphi(x, y)$.

Representing the coordinates in complex form, $z = x + jy$, enables us to set up the function of these coordinates as the function of a complex variable:

$$w(z) = u + jv \tag{5.64}$$

Both the real and the imaginary components of the complex number w certainly represent the functions of two coordinates x and y, i.e. $u = u(x, y)$ and $v = v(x, y)$.

Assume $w = z^2$. This means that $u + jv = (x + jy)^2 = x^2 - y^2 + 2jxy$. Therefore, $u = x^2 - y^2$ and $v = 2xy$.

Those functions of w which are known as analytic functions have definite derivatives

$$dw/dz = \lim_{\Delta z \to 0} \{[w(z + \Delta z) - w(z)]/\Delta z\} \tag{5.65}$$

that are independent of the manner in which the intercept Δz near the observation point z is chosen and what path it takes to approach zero.

This holds for derivatives of any order; in other words, for these functions we have

$$\begin{aligned} dw/dz &= \partial w/\partial x = \partial w/j\partial y \\ d^2w/dz^2 &= \partial^2 w/\partial x^2 = -\partial^2 w/\partial y^2 \end{aligned} \tag{5.66}$$

Separating w into the real and the imaginary parts in second-order derivative expressions (5.66) yields

$$\partial^2 u/\partial x^2 + j(\partial^2 v/\partial x^2) = -\partial^2 u/\partial y^2 - j(\partial^2 v/\partial y^2) \tag{5.67}$$

Expression (5.67) shows that both $u(x, y)$ and $v(x, y)$ satisfy Laplace's equation:

$$\begin{aligned} \nabla^2 u &= \partial^2 u/\partial x^2 + \partial^2 u/\partial y^2 = 0 \\ \nabla^2 v &= \partial^2 v/\partial x^2 + \partial^2 v/\partial y^2 = 0 \end{aligned} \tag{5.68}$$

This remarkable result allows us to consider any of the functions of u or v as the function of potential or the function of flux, for example, to take u = constant as the lines of equal potential and v = constant as the lines of flux normal to the equipotentials.

That the lines u = constant are normal to the lines v = constant can readily be proved considering, for example, the scalar product of two gradients:

$$\operatorname{grad} u \cdot \operatorname{grad} v = (\partial u/\partial x)(\partial v/\partial x) + (\partial u/\partial y)(\partial v/\partial y)$$

It is assumed to be obvious that $\operatorname{grad} f = (\partial f/\partial x)\mathbf{e}_x + (\partial f/\partial y)\mathbf{e}_y$ for each of the gradients, and the scalar product is the sum of the products of similar components. But according to the first equality of (5.66)

$$\partial u/\partial x + j(\partial v/\partial x) = -j(\partial u/\partial y) + \partial v/\partial y$$

or

$$-\partial v/\partial x = \partial u/\partial y, \quad \partial v/\partial y = \partial u/\partial x \tag{5.69}$$

After substituting (5.69) into the scalar product of gradients, we obtain

$$\operatorname{grad} u \cdot \operatorname{grad} v = 0 \tag{5.70}$$

This is exactly the expression of orthogonality of the lines u = constant and v = constant.

As an example, consider the function

$$w = u + jv = \tau' \ln[(z - a)/(z + a)] \tag{5.71}$$

where the complex quantity $z = x + jy$ determines the point on the plane x, y.

It is easy to see that this function describes the field of two charged axes represented by points $z = -a$ (positive charge τ) and $z = +a$ (negative charge $-\tau$). Here Re $w = u$ defines the potential at point z (Fig. 5.13).

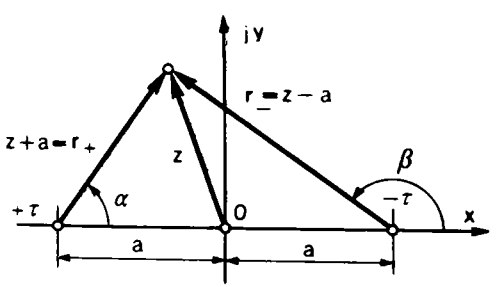


Fig. 5.13

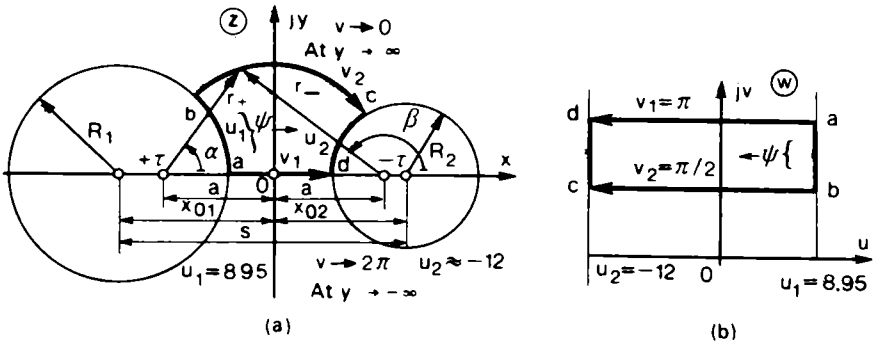


Fig. 5.14

Indeed,

$$z + a = r_+ \angle \alpha \quad \text{and} \quad z - a = r_- \angle \beta$$

are the vectors of which the first is drawn from the positive axis $+\tau$ to the observation point z at an angle α to the horizontal, and the second from the point $-\tau$ at an angle β to the same horizontal.

In this case,

$$w = u + jv = \tau' \ln(r_-/r_+) + j\tau'(\beta - \alpha) \tag{5.72}$$

at

$$u = \tau' \ln(r_-/r_+) \quad \text{and} \quad v = \tau'(\beta - \alpha) \tag{5.73}$$

The equation u = constant at r_-/r_+ = constant is already familiar from Sec. 4.1 (see Fig. 4.3). The equation v = constant is the equation of flux lines or field lines normal to equipotential lines.

Equations (5.73) are the equations for the field of two charged axes, or the equations for the field between two parallel round cylindrical wires whose surfaces coincide with equipotential surfaces u_1 and u_2 (Fig. 5.14*a*). This graphical construction of the field may be regarded as the representation (mapping) on the plane $z = x + jy$ of a simple uniform field which has equipotentials (u = constant) and flux lines (v = constant) on the plane $w = u + jv$ that coincide with cartesian coordinates (Fig. 5.14*b*).

In the general case, the representation of the lines of a uniform field, $w = u + jv$, on the plane $z = x + jy$ can be expressed by any analytic function. Such a representation of $w = f(z)$ certainly satisfies Laplace's equations ($\nabla^2 u = 0$ and $\nabla^2 v = 0$) and displays a very important feature, namely, it is conformal. This means that the transformation here is such that all orthogonal lines remain orthogonal in mapping and any sufficiently small geometric figures remain similar. Fig. 5.14 shows how the rectilinear rectangle in the plane $w = u + jv$ transforms into a curvilinear rectangle whose angles between the lines u and v at the vertexes a, b, c, and d remain square. The conformal representation is unique: each line u or v in the w plane uniquely defines the corresponding line u or v in the z plane. The conclusions follow from the fact that all analytic functions of a complex variable z, i.e. $w = u + jv = f(z)$, satisfy Laplace's equation (5.68).

The above analysis explains why the field calculation method based on the use of the functions of a complex variable received the name conformal representation.

Electric flux. A flux passes between any two lines v_1 and v_2, such as

$$\Psi = (v_1 - v_2)\varepsilon_0\varepsilon \tag{5.74}$$

on condition that a homogeneous medium between the electrodes has a permittivity ε (the flux being determined per unit depth, i.e. per unit length of the parallel electrodes).

In defining any flux (electric current included), we should first specify a positive direction of the flux: formulation (5.74) assumes that the positive flux passes from left to right if the observer moves along the lines drawn from v_1 to v_2 (see Fig. 5.14 and Fig. 5.15).

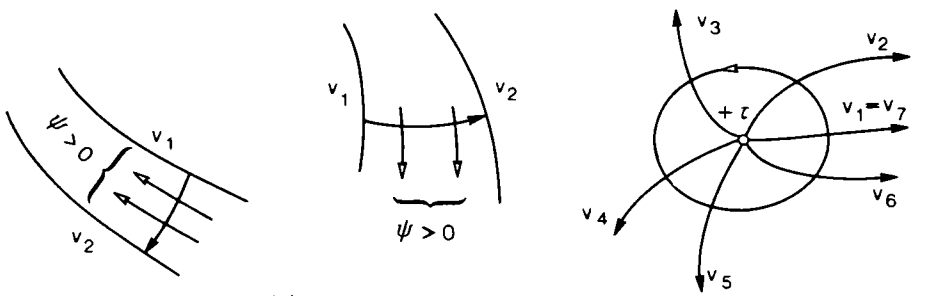


Fig. 5.15

Note that for the specified condition, tracing around the closed path must be in an anticlockwise direction; according to the rule accepted here, the positive flux $\Psi > 0$ will then correspond to the positive charge τ enclosed by the loop (Fig. 5.15*b*).

The flux Ψ bounded by the lines v_1 and v_2 (see Fig. 5.14*a*) is expressed in a representative scale as the difference

$$v_1 - v_2 = m[(\beta_1 - \alpha_1) - (\beta_2 - \alpha_2)] = \Psi$$

Here subscripts 1 and 2 correspond to the angles formed by radii r_+ and r_- sliding about the lines v_1 and v_2. In the map of Fig. 5.14, $\beta_1 - \alpha_1 = \pi - 0$ and $\beta_2 - \alpha_2 = \pi/2$.

It will be worthwhile for the student to try to construct the flux lines for the two-wire line of Fig. 4.3 in such a manner that 1/12 part of the entire flux leaving the wire can flow between each pair of neighbor flux lines. It is obvious from simple geometric considerations that the locus of the line $\beta - \alpha =$ constant is the circumference.

It may also be useful to analyze independently a few functions to make sure, for example, that the real part of the equation

$$w = \tau' \ln[\sin(\pi z/a)] \tag{5.75}$$

expresses the equipotentials of a number of negatively charged axes (if $\tau' > 0$) located at points $z = na$, given any integer n. Superposing the constant component

$w_0 = jz$ on the field under examination gives a field that is similar to the field near a triode grid.

The sum of the field given by (5.75) and the field of the same axes but opposite in direction and displaced by $z_0 = a/2$,

$$w = \tau' \{\ln[\sin(\pi z/a)] - \ln[\cos(\pi z/a)]\} = \tau' \ln[\tan(\pi z/a)] \tag{5.76}$$

expresses the field of the grid of sign-changing axes.

Field strength expression in complex form. Let u be a potential. The field strength is then

$$\mathbf{E} = -\text{grad}\, u = -(\partial u/\partial x)\mathbf{e}_x - (\partial u/\partial y)\mathbf{e}_y \tag{5.77}$$

The field is thus equal to the derivative of the conjugate function

$$\mathbf{E} = -dw^*/dz = -\partial u/\partial x - j\partial u/\partial y \tag{5.78}$$

This is easy to prove. Indeed, by (5.66)

$$dw/dz = \partial u/\partial x + j\partial v/\partial x$$

Substituting $\partial v/\partial x$ from (5.69) into the right side of the expression gives

$$dw/dz = \partial u/\partial x - j\partial u/\partial y$$

Transition to the conjugate number w^* is equivalent to the change of sign of the imaginary component, which completes the proof of (5.78).

For the functions given above, the field of two axes is

$$\mathbf{E} = -dw^*/dz = \tau' \{2a/[(x - jy)^2 - a^2]\} \tag{5.79}$$

5.9. Van-der-Pauw's Theorem

This theorem uses the general features of the equations for plane-parallel potential fields and leads to simple formulas for evaluating conductivity.

As known, electric conductivity is defined by Ohm's law:

$$\sigma = l/RS \tag{5.80}$$

where S and l are the cross-sectional area and the length of a cylindrical specimen; and R is the resistance of the specimen along which a current flows. Cutting out test specimens presents inconvenience, but the main drawback is that the resistance at the contact between the end face terminal and the test bar (the resistance being indefinite and uncontrollable) may greatly affect the measurement results. This is exactly the reason why Thomson replaced Wheatstone's simple bridge by a more elaborate circuit. The effect of contact resistance can be obviated by converting from the measurement of input resistance

$$R = U_{12}/I_{12} \tag{5.81}$$

to the measurement of transfer resistance defined as the ratio of the voltage between the nodes m and n to current when the current enters at the node s and leaves at the node t:

$$R_{(mn)/(st)} = U_{mn}/I_{st}$$

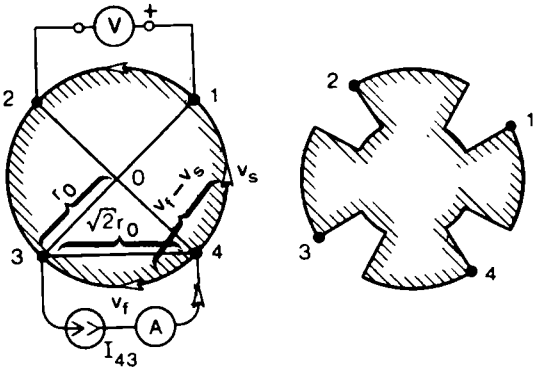


Fig. 5.16

The ratio between the readings on the voltmeter and on the ammeter connected in the circuit of Fig. 5.16*a* is equal to

$$R_{(12)(43)} = U_{12} / I_{43} \tag{5.82}$$

The subscripts here imply that under measurement is the potential difference between electrodes (nodes, or points) *1* and *2*, with the current applied to electrode *4* and drawn from electrode *3* (Fig. 5.16*a*). This concept of resistance measurement lies at the basis of the method described below.

By van-der-Pauw's theorem the conductivity is expressed as the equality

$$e^{-\pi h R' \sigma} + e^{-\pi h R'' \sigma} = 1 \tag{5.83}$$

The formula is suitable for evaluating the conductivity σ of thin sheets of thickness h or electrolytes spread over on a flat horizontal bottom of a nonconducting vessel in a uniform thin layer, etc. The sheet of interest can be shaped to any desired form, either symmetrical (see Fig. 5.16) or asymmetrical (Fig. 5.17). The same also holds for the vertical walls of a tank with an electrolyte (for the generating line of a cylindrical body). The specimen under test has four narrow electrodes (cylinder generatrices) numbered in succession and attached to it around its periphery. Two

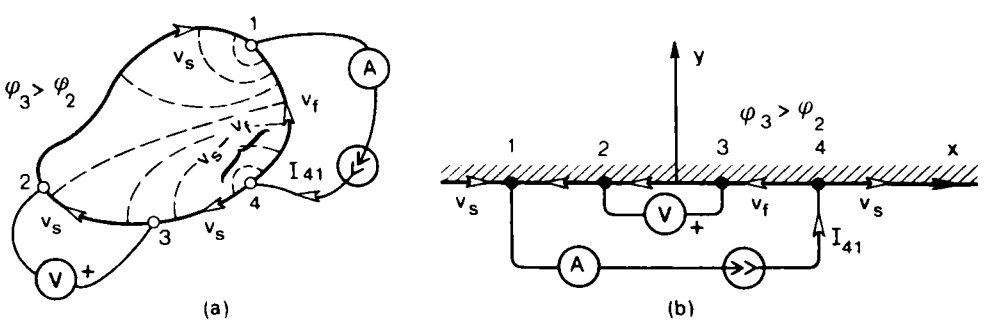


Fig. 5.17

electrodes serve to feed current to the loop and the other two for the measurement of voltage. The effect of symmetry makes itself felt in measurements only if the location of points *1*, *2*, *3*, and *4* is properly chosen. The resistances entering into (5.83) are expressed as

$$R' = U_{12}/I_{43} \quad \text{and} \quad R'' = U_{32}/I_{41} \tag{5.84}$$

Figure 5.16*a* shows the circuit with a voltmeter and ammeter for measuring R', and Fig. 5.17*a* the circuit for measuring R''.

The proof of the theorem rests on the following. In principle, an infinite half-plane (Fig. 5.17*b*) can always be represented on the finite piece of the plane (Fig. 5.17*a*) by transferring all the points on one boundary (on the entire x axis from $-\infty$ to $+\infty$) to the points on the other boundary using the requisite rule of mapping. Here, the identically marked points have the same values (the same coordinates) of current (flux) and potential (v, $u = \varphi$), because it is this feature that determines their identical marking. The whole current flowing from electrode *4* is bounded by two lines of current (flux) v_s and v_f that extend along the boundaries of the plane; the other two points lie just on one of the lines. The difference between the boundary values of current represents the total current that spreads over the specimen. The invariance of φ and v at respective points, whatever the method of transformation used, allows us to assert that the transfer resistance is invariable in conformal representation, i.e. representation by the functions of a complex variable.

Let us now take a look at the half-plane $y > 0$ having at its boundary a successive row of points *1*, *2*, *3*, and *4* spaced at a distance a, b, and c from each other (Fig. 5.18) and evaluate the transfer resistance R' in (5.84). We will use the known formula (4.3) for the potential in the field of two oppositely charged axes. Replacing $\varepsilon\varepsilon_0$ by σ and τ by $2I_{43}/h$ (precisely by $2I$ because the current I_{43} now spreads only on the half of the plane), we obtain for the circuit of Fig. 5.18*a*:

$$\begin{aligned} U_{12} &= \varphi_1 - \varphi_2 = (I_{43}/h\pi\sigma) \ln (r_{13}r_{24}/r_{14}r_{23}) \\ &= (I_{43}/h\pi\sigma) \ln \{[(a + b)(b + c)]/[(a + b + c)b]\} \end{aligned} \tag{5.85}$$

whence

$$R' = (1/h\pi\sigma) \ln \{[(a + b)(b + c)]/[(a + b + c)b]\} \tag{5.86}$$

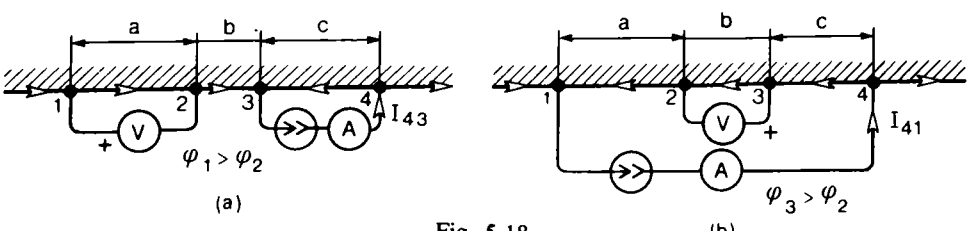


Fig. 5.18

The other transfer resistance R'' is found for the connections shown in Fig. 5.18*b*. The voltage is given by

$$U_{32} = \varphi_3 - \varphi_2 = (I_{41}/h\pi\sigma) \ln [(a + b)(b + c)/ca] \tag{5.87}$$

whence

$$R'' = (1/h\pi\sigma) \ln[(a + b)(b + c)/ca] \tag{5.88}$$

Multiplying R' and R'' by $-h\pi\sigma$, raising e to a respective power, and summing up, we get

$$e^{-R'h\pi\sigma} + e^{-R''h\pi\sigma} = [(a + b + c)b + ca]/[(a + b)(b + c)] \tag{5.89}$$

On performing elementary manipulations we find that the numerator and denominator of the fraction are equal. This proves the validity of formula (5.83) for the upper half-plane. Note that both resistances are positive ($R' > 0$ and $R'' > 0$). Reverse connection of the terminals of the current source and voltmeter leads to a change of the sign of transfer resistances. As mentioned earlier, however, the resistance is invariable in conformal representation and, hence, formula (5.83) is applicable to a plate (cylinder) having any shape of the boundary (periphery).

Of particular interest is the version of (5.83) for symmetric configurations (see Fig. 5.16); the measurements made on these shapes permit us to state that $R' = R''$. In this case, by (5.83)

$$2e^{-R'h\pi\sigma} = 1$$

and, hence,

$$\sigma = (\ln 2)/\pi hR' = 0.220\,6/hR' \tag{5.90}$$

It should be noted that in the specimens shaped as the ones shown in Fig. 5.16*b*, the effect of the transfer resistance of input terminals is negligible since the main voltage drop (read on the voltmeter) occurs in the middle than in its "wings"

Chapter 6 Magnetic Fields. Quasistatic and Quasistationary Conditions

6.1. Solution of Magnetostatic Field Equations

In the absence of currents and, hence, the circulation of a magnetic field ($\operatorname{curl}\mathbf{H} = 0$), we can draw a far reaching formal analogy between a magnetic and an electrostatic field, which permits us to speak of magnetostatics or quasistatic magnetic fields. In these conditions,

$$\begin{aligned} \mathbf{H} &= -\operatorname{grad}\varphi_m \sim \mathbf{E} = -\operatorname{grad}\varphi \\ \mathbf{B} &= \mu_0\mathbf{H} + \mu_0\mathbf{M} \sim \mathbf{D} = \varepsilon_0\mathbf{E} + \mathbf{P} \end{aligned} \tag{6.1}$$

if there is a similarity between $\mu_0\mathbf{M}$ and $\mathbf{P}$ or $\mathbf{M}$ and $\mathbf{P}/\varepsilon_0$. In the case of linear media,

$$\mathbf{B} = \mu_0\mu\mathbf{H} \sim \mathbf{D} = \varepsilon_0\varepsilon\mathbf{E} \tag{6.2}$$

Since $\operatorname{div}\mathbf{B} = 0$, this condition in conjunction with (6.1) allows us to apply Laplace's equation to magnetic fields:

$$\nabla^2\varphi_m = 0 \tag{6.3}$$

But the analogy goes much deeper. For the fields under examination the boundary

conditions at the interface between two media *1* and *2* are similar too:

$$\begin{aligned} H_{1t} = H_{2t} &\sim E_{1t} = E_{2t} \\ B_{1n} = B_{2n} &\sim D_{1n} = D_{2n} \end{aligned} \tag{6.4}$$

which follows from equalities $\operatorname{div}\mathbf{B} = 0$ and $\operatorname{curl}\mathbf{H} = 0$.

There are also important differences between these fields along with the formal similarity. For the conditions being considered, neither magnetic conductors nor magnetic shields exist. True, when a body of, say, $\mu \geqslant 1\,000$ makes contact with the air, the ferromagnet surface can always be regarded as an equipotential, i.e. as the surface of a conductor in electrostatics. Reasoning from the analogy of Maxwell's equations

$$\operatorname{curl}\mathbf{E} = -\partial\mathbf{B}/\partial t \quad \text{and} \quad \operatorname{curl}\mathbf{H} = \mathbf{J}_t \tag{6.5}$$

we may assume that $-\partial\mathbf{B}/\partial t$ is the magnetic flux density.

The formal similarity explained above is of importance because it enables us to apply the solutions found for concrete cases of electrostatic fields to magnetic fields, providing the conditions are the same.

Sphere of permeability μ in an external uniform field. Assuming the permeability of the environment to be equal to unity (a nonpolarizable medium) and analyzing exactly as in Sec. 5.3, we obtain the expressions similar to (5.32) and (5.33):

$$\varphi_{m,e} = -H_0(1 - \eta a^3/R^3)R\cos\theta \tag{6.6a}$$

where $\eta = (\mu - 1)/(\mu + 2)$ and

$$\varphi_{m,i} = -H_0[3/(\mu + 2)]R\cos\theta \tag{6.6b}$$

Likewise, for the internal region ($R \leqslant a$),

$$H_i = H_z = -\partial\varphi_{m,i}/\partial z = H_0 3/(\mu + 2) \tag{6.7}$$

where $z = R\cos\theta$ (see Fig. 5.3). Comparing this result with (5.34c), we find that it follows from (5.34c) on replacing E_0 by H_0 and ε by μ.

Ferromagnetic tube in a transverse field. Applying the method of the solution of Laplace's equation (see Ch. 5), we can readily solve the problem for magnetic shielding with a ferromagnetic tube (shell) immersed in an external transverse magnetic field. Let us set the external (undisturbed) field equal to H_0, the external and the internal radius of the tube to r_e and r_i, the permeability of the tube material to μ (in the environment, $\mu = 1$). Using the same method we find that the field in the internal region ($r \leqslant r_i$) is uniform and equal to

$$\mathbf{H} = k\mathbf{H}_0 \tag{6.8}$$

where the shielding ratio

$$\begin{aligned} k &= 4\mu r_e^2/[(\mu + 1)^2 r_e^2 - (\mu - 1)^2 r_i^2] \\ &\approx 4r_e^2/[\mu(r_e^2 - r_i^2)] \end{aligned} \tag{6.9}$$

(approximation applies to a medium with $\mu \gg 1$).

In distinction to the cases considered above, in the magnetic shielding problem we need to write the solutions similar to (5.16) and (5.17) not for two but for three regions ($r \leqslant r_i$, $r_i \leqslant r \leqslant r_e$, $r_e \leqslant r$) and apply the conditions for the interface between two media not to one but to two boundaries.

Field components due to magnetization. The resultant field may be regarded as the sum of the external field and the field due to magnetization:

$$\mathbf{H} = \mathbf{H}_0 + \mathbf{H}_M \tag{6.10}$$

This expression is completely analogous to (5.36) for electric fields. The vector of magnetic induction is also the sum of two vectors:

$$\mathbf{B} = \mathbf{B}_0 + \mathbf{B}_M \tag{6.11}$$

According to (6.7) and (6.10), inside a magnetized sphere surrounded by the air,

$$H_M = -M/3 \tag{6.12}$$

Comparing (6.12) and (5.44), we should keep in mind the analogy between $\mu_0\mathbf{M}$ and $\mathbf{P}$, as shown in (6.1).

Characteristic cases of calculation of the field for a magnetized sphere. Consider three cases.

1. The magnetization is known and equal to M, its source being of no significance. The external field, i.e. the field in the absence of a magnetic sphere is H_0 = constant. In a particular case, $H_0 = 0$.

2. The sphere in a uniform field with $H_0 \neq 0$. The sphere permeability is μ = constant.

3. The magnetization is a function of the field strength inside the sphere, i.e. $M = F(H)$. The external field has H_0 = constant. In a particular case, $H_0 = 0$.

For all the three cases, we restrict ourselves to the determination of H and B inside the sphere. It is of course not difficult to evaluate the field at any point outside the sphere. For this, we should add to H_0 the field due to magnetization, the latter being determined from (5.45) after the substitution of M for P/ε_0.

For case 1, as is seen from the above analysis,

$$\begin{aligned} H &= H_0 - M/3 \\ B &= \mu_0(H_0 + 2M/3) \end{aligned} \tag{6.13}$$

For case 2, the field strength H is given by (6.7). At

$$M = H(\mu - 1) \tag{6.14}$$

the magnetic induction [see (3.103)] is

$$B = \mu_0(H + M) = \mu_0 H_0 3\mu/(\mu + 2) \tag{6.15}$$

For case 3, we use one more equation $M = F(H)$ along with $H = H_0 - M/3$. To determine M, we should solve these equations simultaneously. The solution is usually found graphically, since the function $F(H)$ for ferromagnets is as a rule given in the form of a hysteresis loop. For this, apart from the graph $M = F(H)$ we should draw a line that represents the second equation $H = H_0 - M/3$.

The points where these lines intersect (Fig. 6.1) determine the sought-for magnetization M. As is clear from the figure, a few factors affect M: the shape of a body (slope of the straight line $-M/3$), external field (displacement of the straight line to H_0) and the property of a material (the type of hysteresis loop). What still remains to be done is to choose one of the points of intersection, *1* or *2*. The choice

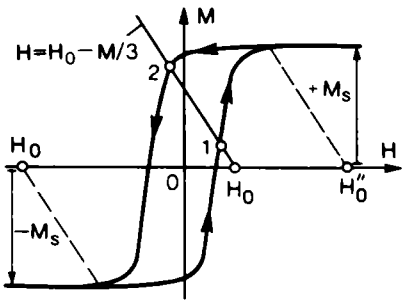


Fig. 6.1

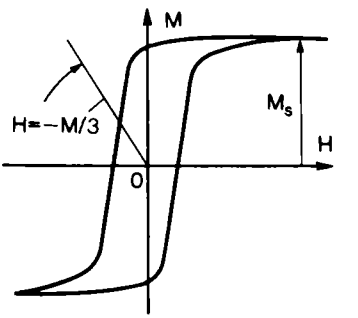


Fig. 6.2

depends on the magnetic history of a specimen, i.e. on the amount and kind of magnetization of the body before its exposure to the field H_0.

For a specimen initially magnetized to saturation $-M_s$, the point of intersection shifts along the ascending branch of the loop as the external field grows, and reaches point *1* at the given value of H_0. For a body previously placed in a strong magnetic field in excess of H_0 and magnetized to $+M_s$, the point of intersection moves down the descending branch as the field decays to H_0, and reaches point *2*. The specimen comes to saturation, $-M_s$ and $+M_s$, in an external field with H_0' and H_0'', as seen from Fig. 6.1.

The way of determining M in a field H_0 of a high positive value which is then brought down to zero is shown in Fig. 6.2.

Knowing M, we can find H and B from formulas (6.13), of course at a definite value of H_0. For the case illustrated in Fig. 6.2, $H_0 = 0$ and $H = -M/3$.

In Figs. 6.1 and 6.2 are shown major loops with tips at $+M_s$ and $-M_s$. They result from a complete cycle of monotonic magnetization from $-M_s$ to $+M_s$ and then back from $+M_s$ to $-M_s$. The *M-H* relation and the effect of the magnetic history of a specimen on this relation can be very complex if partial magnetization cycles that give minor loops take place.

The field strength in a specimen differs greatly from the external field strength. The *M-H* relation for specimens in the form of open magnetic circuits is different from the *M-H* relation for the material proper. The latter (Fig. 6.3) results from the former on shifting all the points laid off on the x axis by

$$H_M = -NM \tag{6.16}$$

where N is the demagnetization (depolarization) factor, equal to 1/3 for a sphere (see Sec. 5.4).

This way of transition from the external field to the field inside a specimen was suggested by Rayleigh and the shift is known as the Rayleigh shift.

Example 6.1. A sphere of radius $a = 0.5$ cm is made of a material that displays a hysteretic property. Its remanent magnetization (i.e. residual magnetization at $H = 0$) is $M_r =$ $= 137$ Oe $= 1\,370$ A/cm $= 1.370 \times 10^5$ A/m and coercive force (at which $M = 0$) is $H_c =$ $= 410$ Oe $= 328$ A/cm $= 3{:}28 \times 10^4$ A/m. The demagnetization curve (the loop branch at

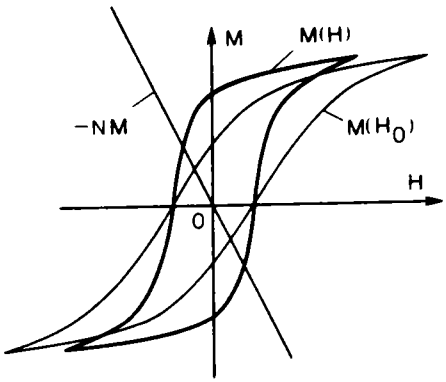


Fig. 6.3

$M \geqslant 0$ and $H \leqslant 0$) features the points:

$-H/H_c$	0.25	0.5	0.75
M/M_r	0.92	0.8	0.55

Find the magnetization of the sphere in the absence of an external field. It is assumed that the sphere was initially magnetized in a strong magnetic field.

Solution. Drawing the straight line $H = H_M = -M/3$ (exactly to scale!), we locate the point of its intersection with the demagnetization curve and thus find that for the data given above $H = -260$ A/cm and $M = 740$ A/cm.

6.2. Demagnetization Factors

Formula (6.16) includes a factor N known as the *demagnetization factor* or the *magnetic coefficient of depolarization*. Formally this factor is introduced into the theory of magnetic fields in the same manner as the depolarization coefficient considered in the theory of potential electric fields (see Sec. 5.4). Its values only depend on the shape of bodies being magnetized. If a body is a spheroid with the z axis of rotation, the values of demagnetization factors are calculated by formulas (5.51) to (5.53). The components due to magnetization are equal to

$$H_{Mx} = -N_x M_x, \quad H_{My} = -N_y M_y, \quad H_{Mz} = -N_z M_z \tag{6.17}$$

On the physical meaning of factors N. The field of an electrically polarized ellipsoid is evaluated by the laws of electrostatics. The internal field, $N\mathbf{P}/\varepsilon_0$, opposing the polarization arises from bound charges $\xi_P = P_n$ on the ellipsoid surface [see Eq. (3.25)]. They produce the field according to Coulomb's law (see Fig. 5.4).

But when we make a similar calculation of the magnetic field in a magnetized body, the following question naturally arises: how can we explain the actual demagnetization caused by fictitious magnetic charges? Considering the question in a formal aspect, we see that everything is correct since the formal competence of magnetostatics is proven. But one certainly has a desire to try to seek an explanation that would be in line reasonably well with the physical understanding of the phenomena in which fictitious magnetic charges do not really participate. Let us follow such a course of reasoning.

The magnetic field in the presence of a magnetized body contains two components given by (6.11):

$$\mathbf{B} = \mathbf{B}_0 + \mathbf{B}_M \tag{6.18}$$

The first arises from the external influences such as the current in a solenoid, magnetization of the bar of an electromagnet, etc., and the second stems from the magnetization of the body proper. As regards an ellipsoid uniformly magnetized parallel to one of its axes, the vector $\mathbf{B}_M$ inside the ellipsoid is parallel to the vector $\mathbf{M}$, the first being related to the second by the proportionality coefficient:

$$B_M = \mu_0 KM$$

or

$$B_M/\mu_0 = KM \tag{6.19}$$

The coefficient K is positive in value and is smaller than unity: $0 \leqslant K \leqslant 1$.

For an elongated ellipsoid magnetized along its body, the coefficient K approaches unity if $(a/c)^2 \to 0$. Indeed, here the equivalent surface current (per centimeter of length) is $I_{eq} = M$ and the magnetic flux density is $B_M = \mu_0 I_{eq} = \mu_0 M$, i.e. such as it is inside a long solenoid. This means that $K = 1$. The coefficient K is equal to zero for an infinite thin plate magnetized in a transverse cross section.

For magnetic induction, the coefficients K may be called magnetization factors. For an ellipsoid,

$$K_x + K_y + K_z = 2 \tag{6.20}$$

For a sphere, $K = 2/3$. For a round cylinder, $K = 1/2$ in transverse magnetization and $K = 1$ in longitudinal magnetization.

Let us now turn to the field strength due to magnetization. By the definition itself,

$$H_M = B_M/\mu_0 - M = (K - 1)M = -(1 - K)M \tag{6.21}$$

In this expression we can readily reveal the demagnetization factor

$$1 \geqslant 1 - K = N \geqslant 0 \tag{6.22}$$

Ellipsoid in an external uniform field; shape anisotropy. Consider a spheroid of permeability μ. Decompose the external field vector $\mathbf{H}_0$ into two components, the component H_z parallel to the axis of rotation and the component H_x normal to this axis:

$$\mathbf{H}_0 = H_{0x}\mathbf{e}_x + H_{0z}\mathbf{e}_z \tag{6.23}$$

The expression defines the chosen system of coordinates (Fig. 6.4).

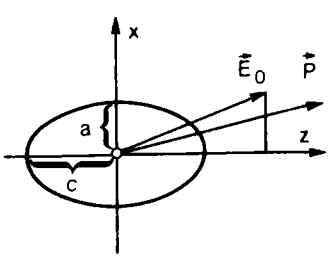


Fig. 6.4

The assumption that the permeability remains constant is equivalent to the statement that the medium shows linear characteristics and allows for superposition. The magnetization of the ellipsoid along the x and z axes can then be written as

$$\begin{aligned} M_x &= k_m H_x = k_m(H_{0x} - N_x M_x) \\ M_z &= k_m(H_{0z} - N_z M_z) \end{aligned} \tag{6.24}$$

In these formulas, the parantheses enclose respectively the xth and zth components of the field **H** inside the magnetized body [see. Eqs. (6.17)]; the factor k_m is the susceptibility equal to $\mu - 1 = k$. Solving (6.24) for magnetization components gives

$$\begin{aligned} M_x &= kH_{0x}/(1 + kN_x) \\ M_z &= kH_{0z}/(1 + kN_z) \end{aligned} \tag{6.25}$$

Because N_x and N_z differ in value, the magnetization vector does not coincide in direction with the external field strength vector (see. Fig. 6.4) and keeps closer to the semimajor c axis. Really, from (6.25) it is obvious that

$$M_x/M_z \neq H_{0x}/H_{0z} \text{ at } N_x \neq N_z \tag{6.26}$$

The medium is assumed to be isotropic, so that its μ and k_m are independent of the direction of magnetization. But we have arrived at inequality (6.26) specific to an anisotropic medium. In the given case, anisotropy stems from the axial variation of the ellipsoid, and this is the *shape anisotropy*.

Demagnetization factor in a magnetic circuit. In Example 3.4 of Sec. 3.3 we considered a simple circuit. It consisted of a toroidal ferromagnetic core of length l_f with a small transverse air gap of length l_a. When no current flows in the toroid coil and the ferromagnet retains the magnetization M, the relation between M and H inside the ferromagnet takes the form

$$H_f = H_m = -Ml_a/(l_a + l_f) \tag{6.27}$$

In this formula [see also formula (d) of Example 3.4] deduced from the circuital law, we easily find the demagnetization factor

$$N = l_a/(l_a + l_f) \tag{6.28}$$

Figure 6.5*a* illustrates the magnetic circuit in a somewhat more general form; in Fig. 6.5*b* is shown the magnetic circuit of a permanent magnet. The magnet and the air gap are assumed to be different in cross section. Besides, the magnetic flux Φ_a across the air gap is taken to be smaller than the flux Φ_m across the average cross section of the magnet on account of a possible branching, or leakage of the flux:

$$\Phi_m = \mu_0(H_m + M)S_m = k_{fl}\Phi_a \tag{6.29}$$

where k_{fl} is the flux coefficient. It is higher than unity; its value is set up from experience or found from calculations.

Applying the circuital law, we can write the equation

$$H_m l_m + k_p H_a l_a = 0 \tag{6.30}$$

where l_m is the length of the permanent magnet.

This equation allows not only for the field strength of the air gap, $H_a l_a$ (magnetic potential difference), but also for the magnetic field strength that falls on the connec-

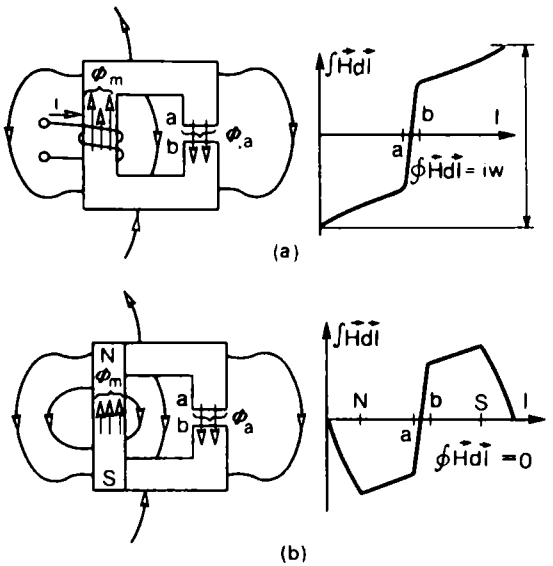


Fig. 6.5

ting branches. The latter field strength is taken into account by the coefficient k_{fl} which is higher than unity. This coefficient shows that the permanent magnet mmf$(-H_m l_m)$ must be higher in magnitude than the magnetic potential difference across the air gap. This is due to the fact that the connecting portion of the magnetic circuit shows a certain resistance and the active length of the magnet is somewhat shorter than its geometric length l_m, at least because the magnetic flux lines deviate from the direction of the vector **M**.

Assuming that

$$H_a = \Phi_a / S_a = \Phi_m / k_{fl} S_a \tag{6.31}$$

we find from the simultaneous solution of the last three equations:

$$H_m = -NM \tag{6.32}$$

where

$$N = R_a / (R_a + \alpha R_m) \tag{6.33}$$

at $\alpha = k_{fl}/k_p$, $R_a = l_a/\mu_0 S_a$, $R_m = l_m/\mu_0 S_m$.

Expression (6.33) simplifies to (6.28) at $S_a = S_m$, $\Phi_m = \Phi_a$ and $k_{fl} = k_p = 1$.

After determining the demagnetization factor, we can apply the method illustrated by Fig. 6.2 and locate the operating point for the material of the magnet, i.e. estimate M and $H = H_m$.

6.3. Method of Images for a Magnetic Field

All the solutions obtained in Sec. 5.7 for an electrostatic field are also applicable to a stationary or quasistationary magnetic field.

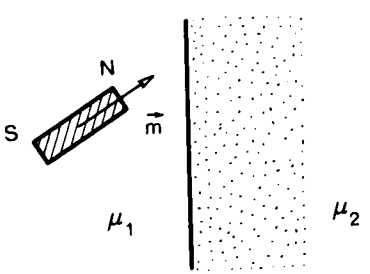


Fig. 6.6

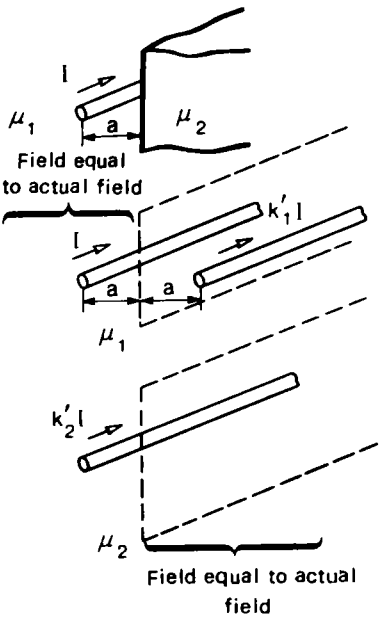


Fig. 6.7

Consider a magnetized bar with a moment **m** that lies in the air ($\mu_1 = 1$) near the flat boundary between the air and a medium of permeability μ_2 (Fig. 6.6).

By the method of images, the field in the first medium should be taken equal to the field of the given bar with moment **m** and the field of its image with moment $k_1\mathbf{m}$ (since each charge has changed k_1 times in imaging, the moment of the charge differs from the moment of its image by a factor of k_1). By analogy to (5.63), we have

$$k_1 = (\mu_1 - \mu_2)/(\mu_1 + \mu_2) \tag{6.34}$$

taking the permeability of both media equal to μ_1. The further calculation is also similar to that in electrostatics.

Another important case of the calculation by the method of images involves a current-carrying conductor at the boundary between two different magnetic media. The field equations and boundary conditions can be satisfied if the calculation is carried out in accordance with the scheme presented in Fig. 6.7.

The field in the first medium ($\mu = \mu_1$) is equal to the field of two currents: the current I in the given wire and the current $k_1'I$ in its image, the permeability of the unbounded medium being μ_1. Here,

$$k_1' = (\mu_2 - \mu_1)/(\mu_1 + \mu_2) \tag{6.35}$$

The field in the second medium ($\mu = \mu_2$) is equal to the field of the actual wire but carrying the current $k_2'I$. Here,

$$k_2' = 2\mu_1/(\mu_1 + \mu_2) \tag{6.36}$$

the entire space being filled with the medium of permeability μ_2.

It is easy to show that the boundary conditions and basic equations are satisfied precisely at these values of currents.

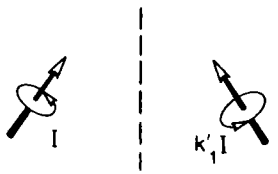


Fig. 6.8

The differences between k_1 and k_1' and also between k_2 and k_2' are due to the transposition of μ_1 and μ_2. The explanation of the fact is as follows. If the moment does not change sign in imaging, the current in the equivalent loop flows in the opposite direction (Fig. 6.8). This explains the reversal of sign in respective coefficients ($k_1' = -k_1$).

6.4. Magnetic Field in Electrical Machines

The calculation of a magnetic field in electrical machines presents an important problem for machine-building engineers. It is usual practice to make certain assumptions and thus simplify the calculations. Of course, the assumptions must be such that they permit the engineer to estimate correctly the parameters and find the ways to improve the design. If the machine length in the axial direction is much greater than the stator and rotor radii, the field may be considered to depend only on two coordinates, r and α, disregarding a small region near the end faces.

The field calculation thus reduces to evaluating the field in the gap between the salient poles of the stator and the smooth rotor (see. Fig. 4.23), excluding the effect of pole teeth between the slots in which coils are laid. The field distribution in the gap enables us to judge of the pattern of the emf being induced, reluctance in the gap, etc. More elaborate calculations can also take into consideration the effect of teeth.

The field in a gap between the round stator and rotor of an electrical machine. The estimate of such a field, even being fairly tentative, is of essential significance. The assumption of the round surface of stator and rotor is a reasonable approximation to the design of synchronous high-speed nonsalient-pole machines (Fig. 6.9). The magnetic field in the gap of such a machine is determined by the currents in the coils installed in special slots, as shown in Fig. 6.10 (for a stator). The slots are cut in the massive body of a cylindrical rotor to receive a dc winding. The slots for an ac winding are punched in sheet-steel laminations which are then stacked to form a stator ring.

Disregarding a steplike transition from slot to slot and from wire to wire, we may assume that the surface current per unit length of the perimeter varies in a simple sinusoidal way:

$$k = K' \cos\alpha + K'' \sin\alpha \tag{6.37}$$

This is precisely the expression for the first harmonic component in the periodic spatial distribution of current over the stator and rotor surface.

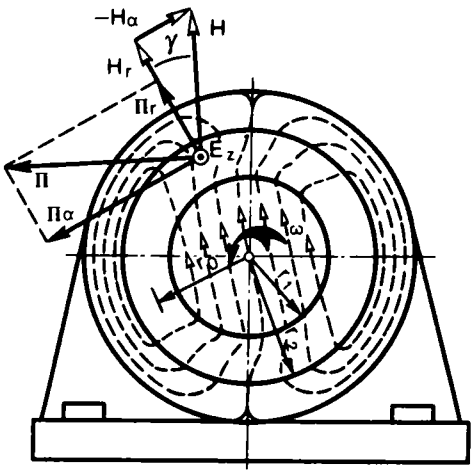


Fig. 6.9

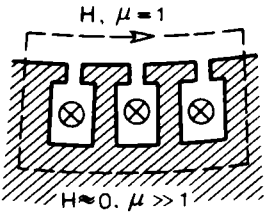
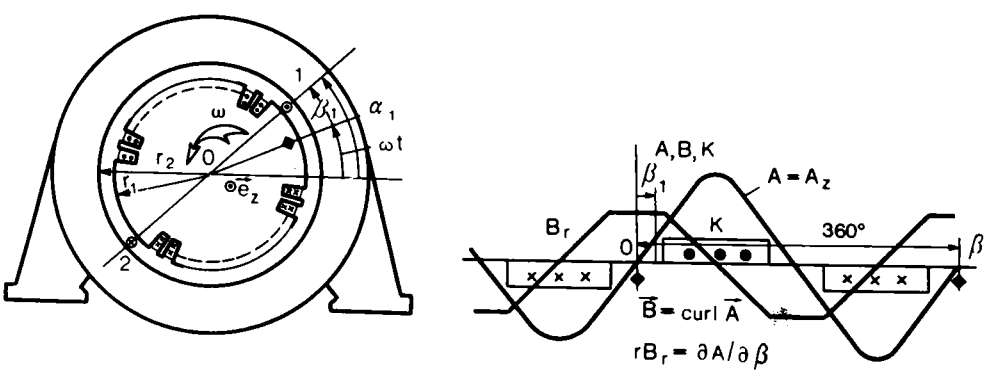


Fig. 6.10

If α is a solid angle, expression (6.37) only holds for magnetic systems with one pair of poles around the entire circumference (if the system has n pairs of poles, α should be replaced by $n\alpha$).

Assuming the permeability of the stator and rotor to be high, $\mu \gg 1$, the field strength may be taken to be different from zero only in the air. The surface current then uniquely defines the tangential component of field strength on the rotor surface ($r = r_1$, $H_\alpha = H_{\alpha 1}$) and stator surface ($r = r_2$, $H_\alpha = H_{\alpha 2}$).

Let the rotor (Fig. 6.11*a*) have a winding in which one half of its coils carry current into the paper, and the other out of the paper, each half being laid in slots around one third of the circumference (120°). The layout of magnet wires on the linearly developed rotor surface is shown in Fig. 6.11*b*; the same figure also illustrates the surface current density $K = K_z$. The relation between $K = K_z$ and angle β quantitatively coincides with the relation between the field-strength tangential component $K_z = H_\alpha$ on the rotor surface ($r = r_1$) and the same angle β. The quadrantal angle β is



(a) 6.11 (b)

counted off from the symmetry plane between the opposite coils of the winding (see Fig. 6.11). The first harmonic in the periodic distribution of the surface current density and the field-strength tangential component on the rotor surface ($r = r_1$) is given by

$$H_{\alpha,1} = K_{z,1} = K_1 \sin\beta$$

In the general case where the choice of the reference axis for counting off the angles β is arbitrary, both the current and field strength can also include a cosine component.

When the rotor runs at a speed ω, the angle of reference in the system of coordinates of the immobile observer is

$$\alpha = \beta + \omega t \tag{6.38}$$

An ordinary synchronous machine has a three-phase winding inserted in the slots of the stator and distributed so that the sinusoidal three-phase current is followed by a magnetic field having on the stator surface ($r = r_2$) a tangential component

$$H_{\alpha,2} = -K_2 \cos\beta = -K_2 \cos(\alpha - \omega t) \tag{6.39a}$$

This notation which excludes the sinusoidal component (and includes only the cosine component) certainly corresponds to a specially chosen reference axis and an origin from which we reckon the angle α and time t. In the β coordinate system rotating together with the rotor, the tangential component is stationary; with a change in angle β this component changes according to a simple sinusoidal rule, independent of time. To the immobile observer (immobile relative to the stator) whose coordinate system is such that α serves as an angular measure, this field appears to be revolving. This means that at a point where the angle α is constant, the field sinusoidally varies with time and the point of a constant phase ($\beta = \alpha - \omega t$ = constant) rotates at an angular velocity $\omega = \partial\alpha/\partial t$.

Assuming that $H_{\alpha,2}$ varies in accordance with (6.39a), we thus define the line and point of reference for β and t which may be inconsistent with the reference axis shown in Fig. 6.11 for counting off the angles of rotor rotation. In a more general case, therefore, the tangential field-strength component on the rotor ($r = r_1$) must be expressed as

$$H_{\alpha,1} = K_1' \cos\beta + K_1'' \sin\beta \tag{6.39b}$$

The two equalities (6.39) are the boundary conditions for the field in the air gap, i.e. in the cylindrical layer confined between the rotor and stator ($r_1 \leqslant r \leqslant r_2$).

In the calculations made earlier we dealt with two kinds of vector functions of r and α, the tangential component of which varies periodically with the angle [see Eq. (5.30b)]. One of these functions is a constant quantity multiplied by $\sin\alpha$ and the other is directly proportional to $\sin\alpha$ but inversely proportional to the square of the radius. The choice of the reference line for α remains arbitrary. The appropriate combination of these functions can satisfy the given boundary conditions. As a matter of fact, we consider here vector functions $\mathbf{H}(r, \alpha)$ and not functions $\mathbf{E}(r, \alpha)$. But both these functions, including $\mathbf{H}$ and $\mathbf{E}$ themselves which are analogous to each other, comply with the requirement for the absence of curls and sources, i.e. the functions $\mathbf{H}(r, \alpha)$ satisfy the equations of a magnetic field in an air gap:

$$\operatorname{div}\mathbf{H} = 0 \text{ and } \operatorname{curl}\mathbf{H} = 0$$

In the functions $\mathbf{H}(\alpha, r)$ being considered the choice of constants can be made such as to satisfy boundary conditions (6.39). The solution then has the form

$$\rho^2 H_\alpha = (K_1' r_1^2 h_2 + K_2 r_2^2 h_1)\cos\alpha + K_1'' r_1^2 h_2 \sin\alpha \tag{6.40}$$

and

$$\rho^2 H_r = K_1'' r_1^2 h_4 \cos\alpha - (K_1' r_1^2 h_4 + K_2 r_2^2 h_3)\sin\alpha \tag{6.41}$$

where

$$\rho^2 = r_2^2 - r_1^2,\ h_{1,3} = r_1^2/r^2 \mp 1,\ h_{2,4} = r_2^2/r^2 \mp 1$$

As seen from these formulas, *the radial component* that varies in the air gap as a function of the angle *is certainly followed by a tangential component.*

It is apparent from (6.40) that the tangential components caused by the rotor winding vanish as the distance from the rotor to the stator increases (the factor $h_2 = r_2^2/r^2 - 1$ goes to zero). In a similar manner, the effect of the stator winding current on the tangential field component decays farther away from the stator. The boundary conditions are thus met. The radial components near the stator and rotor surfaces [see. Eq. (6.41)] result from the currents in both windings. The way of using formulas (6.40) and (6.41) will be shown in Example 7.4, Sec. 7.3.

The calculation performed here is also of interest as an example of the solution for the field equations with the given tangential components of a field at the boundary (rather than with the normal components as is the case for Neumann's problem formulated in Sec. 5.1).

6.5. Vector Potential for the Magnetic Field in an Electrical Machine

It is of importance to know the vector potential at the points of location of conductors where the emf is induced while the machine is running. In stationary conductors, the induced electric field is given by

$$E = -\partial A/\partial t \tag{6.42}$$

Let the rotor of Fig. 6.11*a* carry a dc winding (slip rings and brushes being omitted for simplicity). The rotor revolves at a speed ω. The stator conductors *1* and *2* in which we wish to specify the emf form a loop, i.e. a turn of the stator winding. Assume the first conductor lies on the radius making an angle α_1 with the fixed reference axis for angle measurement. The same radius forms an alternate angle

$$\beta_1 = \alpha_1 - \omega t \tag{6.43}$$

with the fixed rotor diameter indicated in Fig. 6.11*a* along with other quantities.

Figure 6.11*b* shows the values of the following quantities as functions of the angle β: the radial component of magnetic induction, B_r (at a certain fixed radius, for example, on the stator surface, $r = r_2$), the vector potential $A = A_z$, and current in the rotor winding, the current being represented as the equivalent surface current density $K = K_z$.

The sketch of Fig. 6.11*b* is drawn on the assumption that each half of the winding is arranged around 2/3 of the circumference; the distribution of B_r as a function of β

is nearly sinusoidal (the third harmonic is not present, and the fifth has an amplitude accounting for 4% that of the first). The function $A(\beta)$ then also closely resembles a sinusoid. The graph so plotted does not take into account the effect of slots and teeth.

If the function $A(\beta)$ for any point is known, then, assuming the rotor runs at an angular velocity $\omega = -\partial\beta/\partial t$, the time-varying vector potential at a fixed point in space (α is constant) is given by

$$-\partial A/\partial t = -(\partial A/\partial\beta)(\partial\beta/\partial t) = (\partial A/\partial\beta)\omega$$

The sought-for electric field induced in the fixed conductor is equal to the same value:

$$E = -\partial A/\partial t = (\partial A/\partial\beta)\omega \tag{6.44}$$

It will be recalled once again that the field is plane-parallel, i.e. $A = A_z$ and $E = E_z$.

If a fixed conductor lies in the gap between the smooth surfaces of the stator and rotor (see Fig. 6.11*a*), then the electric field induced in the conductor can be expressed as

$$E = E_z = \omega r B_r = \omega r \mu_0 H_r \tag{6.45}$$

This formula directly follows from (6.44) since

$$rB_r = \partial A_z/\partial\beta \tag{6.46}$$

considering that $\mathbf{B} = \operatorname{curl}\mathbf{A}$.

Formula (6.45) is very popular and is often derived on the basis of simple considerations of wire motion in the magnetic field. But the formula is applicable only for the region of the air gap between two round cylindrical surfaces of the stator and rotor. Indeed, B_r inside the slot (Fig. 6.12) proves negligible, and the actual field being induced is hundreds and thousands of times as high as that found from (6.45). Formula (6.45) is also impracticable if the wire that lies between the smooth surfaces of the rotor and stator is of iron rather than of copper. Because of its high permeability, the iron wire will have twice the radial magnetic-induction component of the copper wire under similar conditions. Nevertheless, the field induced in either wire is the same:

$$E_{\mathrm{Fe}} = E_{\mathrm{Cu}} \text{ at } B_{r,\mathrm{Fe}} = 2B_{r,\mathrm{Cu}} \tag{6.47}$$

Figure 6.12 illustrates a detail of the stator with deep slots for conductors. The winding of the rotor (see Fig. 6.11) creates a high field. The slots affect this field making it certainly nonuniform, as is apparent from Fig. 6.12, where the graphs plotted take into consideration only the effect of stator slots.

Assume that in the region of the highest field, the radial component is equal to 0.9 T (9 kG) and remains constant on a certain interval of the angle β. Because of a high permeability of steel teeth ($\mu \approx 2000$) the magnetic flux mainly bypasses the air slots; the magnetic induction in the slots is a factor of μ lower than that in steel teeth, where it is approximately twice as high as the mean magnetic induction in the slot. The induction inside the slot may be taken equal to about 0.001 T (10 G).

The distribution of vector potential $A = A_z$ in the stator at $r = r_2$ as a function of the angle β or $\alpha = \beta + \omega t$ is shown in Fig. 6.12 for the three successive moments of

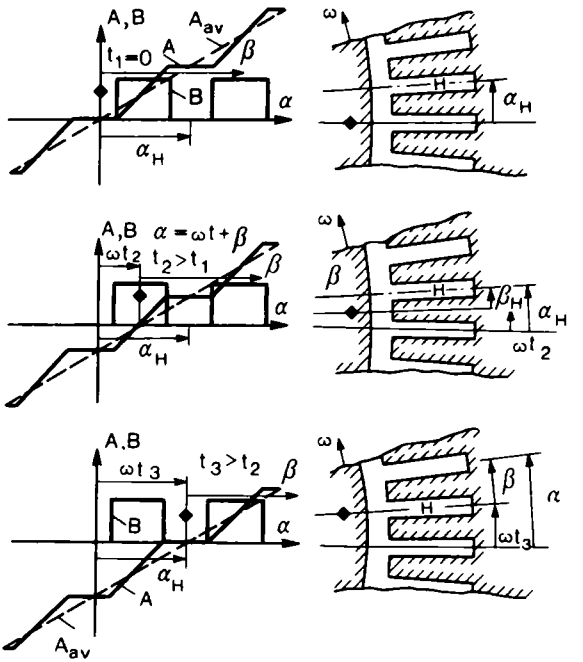


Fig. 6.12

time, $t_1 = 0$, t_2, and t_3. The straight line represents the dependence of the average potential A_{av} in the absence of the effect of teeth and slots; the slope of this line is proportional to the average value of B_r, as follows from formula (6.46) applied to the average values of A and B_r.

But the slope of the line $A(\beta)$ varies sharply (by a factor of 2000!) in the portions representative of steel and the slot, as is clear from formula (6.46) used for the calculation at r = constant. In the scale of Fig. 6.12, the lines of the vector potential for slots are just horizontal intercepts (A is almost constant at $B_r \approx 0$), but the slope of the lines $A(\beta)$ in the portions representative of steel is greater than the average slope [1]. Although it remains almost invariable at various points of one slot (i.e. does not practically depend each instant on spatial coordinates), the vector potential smoothly decays with time as the rotor goes on revolving (Fig. 6.13). This is obvious from the comparison of the values of the vector potential at any point, for example, at point H in Fig. 6.12 for various moments of time, t_1, t_2, t_3, etc. Thus, in our case, $\partial A/\partial t = \partial A_{av}/\partial t$, though $A \neq A_{av}$. Here formula (6.45) proves valid, too, if we substitute the specified values of $B_{r,\,av}$ (or $H_{r,\,av}$) for B_r characteristic of the actual magnetic field.

As for a closed slot, the process of inducing the electric field in the wires inserted in this slot is similar to the process occurring in transformer windings that can be

[1] The vector potential is taken to be zero on the rotor axis. This is equivalent to the zero potential on the rotor surface at $\beta = 0$ (see Fig. 6.11).

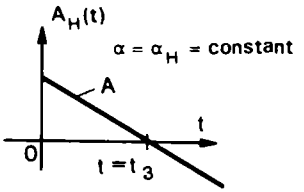


Fig. 6.13

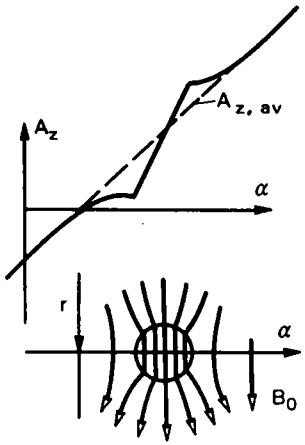


Fig. 6.14

disposed in the region where magnetic induction does not exist ($B = 0$). For this reason the component $-\partial A/\partial t$ of the induced electric field is referred to as a transformer component.

The seemingly contradictory conclusion expressed by (6.47) is explained in a similar manner. Fig. 6.14 shows how the vector potential $A = A_z$ changes along the line r = constant that passes through the wire axis, the external field vector $\mathbf{B}_0$ being directed along r. The same figure also illustrates the field outside and inside the iron wire normal to the plane of the paper. From the analysis of the field pattern (showing a twofold increase in induction inside the wire) and the curve $A(\alpha) = A_z(\alpha)$, it is easy to see that the vector potential inside the wire must change; it changes twice as fast as the average potential whose variations are shown by a dashed line. The vector potential changes little near the iron wire since the magnetic induction in close vicinity to the wire is a factor of μ as low as inside the wire (the induction in question is that along the line r = constant). But as the rotor continues rotating, the slope of the steep portion of the line $A(\alpha)$ changes so that the midpoint of this line slides along the average straight line $A_{av}(\alpha)$. For any point inside the conductor, the derivative $-\partial A/\partial t$ will be the same. It is equal to the rate of change of the average vector potential:

$$-\partial A/\partial t = -\partial A_{av}/\partial t \tag{6.48}$$

The function $A(r, \beta)$ is amenable to calculations only in rare cases. The attempts to take into account the effect of rotor tips and coil distribution inside a slot lead to additional difficulties. To achieve a success, the method of field modeling in an electrolytic tank can be used in case of plane-meridian and plane-parallel fields if $A = A_z$ and the equation for vector potential is nothing else but Poisson's equation:

$$\nabla^2 A = -J_z/\mu\mu_0$$

Chapter 7 ENERGY AND FORCE

7.1. General

The transmission and conversion of energy is of much practical importance in electrical engineering. Energy transmission and conversion commonly play a significant part also in cases where the purpose of a technical device is to transmit, convert, and receive information.

In its initial stages, electrodynamics developed as a science dealing with forces and motion of mechanical bodies due to electromagnetic processes (in a way, that was how Ampère understood this science, who was the first to introduce the word electrodynamics). At that time, the formulations containing only integral quantities and parameters often played a basic role.

Electric and magnetic energy. In the conditions of linearity, the electric energy of a charged capacitor and the magnetic energy of a coil carrying a current can be expressed in terms of voltage, current, charge, and magnetic flux (flux linkage)

$$W_e = u^2C/2 = uq/2$$
$$W_m = i^2L/2 = i\Psi/2 \tag{7.1}$$

without resorting to the concepts of field theory and disregarding the problem on the space localization of this energy.

These expressions can be extended to cover a system of charged bodies and a system of current loops:

$$W_e = \varphi_k q_k/2$$
$$W_m = \Psi_k i_k/2 \tag{7.1a}$$

In these formulas, the summation is taken with respect to the terms having the same index, though the sign of the sum is not given here for the simplicity of notation. In the first formula, the summation is made over all the charges q_k. This formula can be derived from the general expression for electric field energy (see Sec. 7.2) or obtained from formula (7.1) by applying it to the system of plates of a number of capacitors (direct capacitances). In the second formula, the summation is taken with respect to all the currents i_k in the loop k being linked by the flux Ψ_k. The second formula is similar to the first, the only difference being that the vector potential $\mathbf{A}_k$ in the former is replaced by the integral

$$\Psi_k = \oint_{ik} \mathbf{A} d\mathbf{l}$$

because the vector potential at various points of the loop k is different.

The expressions containing the integral parameters of circuits find general use, particularly in circuit theory:

$$W_e = \varphi_k \varphi_l C_{kl}/2$$
$$W_m = i_k i_l L_{kl}/2 \tag{7.2}$$

Here the summation is taken over the k terms and l terms (from 1 to n), i.e. for each k the summation is extended over all l.

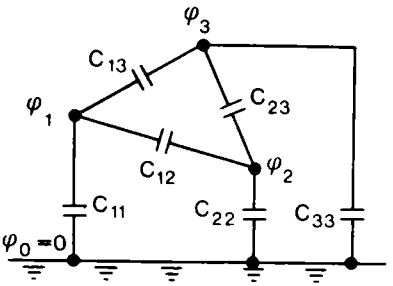


Fig. 7.1

In a particular case, for two loops

$$W_m = (i_1^2 L_{11} + i_2^2 L_{22})/2 + i_1 i_2 L_{12} \tag{7.3}$$

where $L_{11} = L_1$, $L_{22} = L_2$, and $L_{12} = L_{21} = M_{12}$.

The derivation of this formula for two loops is given in Sec. 4.2 [see. Eq. (4.75)]. The derivation can be extended over n loops.

The first of the two formulas (7.2) for electric energy contains the coefficient of self-capacitance, c_{mm}, and the coefficient of mutual capacitance, c_{mn}, which are formally analogous to those of self-inductance and mutual inductance. These coefficients are not amenable to physical interpretation in such an illustrative way as direct capacitances presented in Fig. 7.1. This is the reason why the coefficients of capacitance are not popular. Applying the first of the two formulas (7.1) to all capacitances that constitute the system of capacitor plates, and expressing each of the voltages across the "capacitors" as a potential difference, we easily obtain the first of the two formulas (7.2). All the coefficients c_{mn} are found to be uniquely related to the direct capacitances. By performing these operations for the three pairs of plates shown in Fig. 7.1, we have

$$c_{11} = C_{11} + C_{12} + C_{13}$$
$$c_{22} = C_{21} + C_{22} + C_{23} \ldots$$
$$c_{12} = -C_{12}, c_{23} = -C_{23} \ldots$$

For the energy transmitted along wires, the power can be expressed in terms of voltage and current

$$p = ui$$

ignoring the question of where and along what path the transfer of energy takes place from the source to the consumer.

All the formulas given above contain only integral quantities such as u, i, q, and Ψ, while the quantities in the explicit form that characterize the field, such as **E**, **D**, **H**, and **B** do not enter into the formulas. From the viewpoint of field theory, however, the processes of energy transmission and conversion in electromagnetic systems should be considered as the processes that occur in the field, are due to the field, and expressed by means of field equations. The energy of electromagnetic systems should also be regarded as the energy of a field.

That the energy of a field indeed exists is also evident from the following: the transfer of energy in electromagnetic systems can occur without the use of wires

(radio), and the electromagnetic field itself can appear in the absence of both charges and currents. The field energy and its transfer, i.e. the flow of field energy, are dealt with in Sec. 7.2.

In the analysis of energy relationships of much significance is Joule's law that shows how the thermal dissipation of electromagnetic energy takes place. The differential form of Joule's law is given by

$$P_0 = J^2/\sigma = E^2\sigma$$

where P_0 is the power related to a unit volume; and σ is the conductivity. It is assumed that the current density is related to the field strength by a simple expression (Ohm's law)

$$\mathbf{J} = \mathbf{E}\sigma$$

Forces in electromagnetic systems. Forces can be defined in terms of energy, disregarding field concepts. For this it is enough to use the expression of energy in terms of integral parameters and apply the law of conservation of energy to the system under examination.

The following energy balance corresponds to the law of conservation of energy:

$$dW_s = dW + dA \tag{7.4}$$

Here dW_s is the energy supplied to the system from the source (generator), excluding the energy transformed into other kinds of energy, for example, into heat, in accordance with Joule's law (i^2rdt); dW is an increment in the electromagnetic energy (field energy); and dA is the mechanical work done by the force being considered that causes an elementary displacement.

In the general case, the element of work is equal to the element of displacement in a given direction times the respective component of force, i.e.,

$$dA = f_\eta d\eta \tag{7.5}$$

It is certain that the balance equation contains the elementary increment in electromagnetic energy, dW, and the enegry dW_s drawn from the source and determined precisely by the displacement $d\eta$:

$$dW = (\partial W/\partial\eta)d\eta \text{ and } dW_s = (\partial W_s/\partial\eta)d\eta$$

From (7.4) it then follows that

$$\partial W_s/\partial\eta = \partial W/\partial\eta + f_\eta \tag{7.6}$$

If the system does not receive energy from the source, then the work can only be done at the expense of the field energy:

$$dA = -dW \tag{7.7}$$

Correspondingly,

$$f_\eta = -\partial W/\partial\eta \tag{7.8}$$

In the above equations the coordinate η and force f_η are generalized coordinates and forces, the product of which is equal to the work given by (7.5). Thus, if η is the angle of rotation, α, then the respective generalized component of force is the mechanical torque T_α. Indeed,

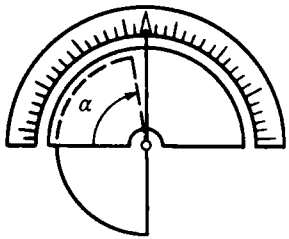


Fig. 7.2

$$f_\eta d\eta = T_\alpha d\alpha = dA \tag{7.9}$$

The simple examples that follow illustrate the use of the relations derived above.

Example 7.1. The electrostatic voltmeter of Fig. 7.2 consists of a row of parallel plates shaped as semicircles and another row of semicircular plates disposed in between. The first system of plates is stationary and the second hangs on an elastic string, can rotate about its axis, and has a mirror or a pointer attached to it to indicate the reading. The torque of the string being twisted is proportional to the angle α. The torque counteracts the rotation of the moving system. All the moving semicircular plates are connected to each other and isolated from the system of stationary interconnected plates. If the plates experience the torque T as a result of interaction of charges, the moving system turns by an angle of α; hence, the work is done on the filament by twisting it, which thus stores a potential energy.

The capacitance of a parallel-plate capacitor is proportional to the area of plates. In this instrument the active area of plates, i.e. the area portion that can be regarded as the surface of plates of a parallel-plate capacitor, is proportional to the angle α. The capacitance of the system is then

$$C = C_0 + k\alpha$$

where k is the proportionality factor chosen from design considerations.

Find the relation between the angle of swing, α, of plates (and the pointer) and the applied voltage.

Solution. As a result of interaction of oppositely charged plates, the moving system experiences a torque T that tends to increase the angle α.

The element of work being done

$$dA = Td\alpha$$

results in the potential energy of the string being twisted that counteracts the system rotation.

Determine the force from energy balance formula (7.6). Assume that the supply source maintains a constant voltage U. From (7.6) we then get

$$f_\alpha = T = \partial W_s/\partial\alpha - \partial W/\partial\alpha$$

The energy delivered by the source at a constant voltage is defined by a charge $dq = UdC$:

$$dW_s = Udq = U^2 dC$$

and therefore

$$\partial W_s/\partial\alpha = U^2 \partial C/\partial\alpha = U^2 k$$

An increment in the field energy at a constant voltage is

$$dW = U^2 dC/2$$

and hence

$$\partial W/\partial\alpha = U^2(\partial C/\partial\alpha)/2 = U^2 k/2$$

It is of interest to note that at a constant voltage (with the energy being replenished by the source), the field energy rises rather than decays.

Finally, we find that

$$f_\alpha = T = U^2 k/2$$

Assuming that the counteracting torque T of the string being twisted is directly proportional to the angle of twist,

$$T = K\alpha$$

we obtain

$$K\alpha = U^2 k/2 \text{ or } \alpha = kU^2/2K$$

where K is the string rigidity.

Example 7.2. Find a force f_x on a crosspiece, or connector (moving element) of the switch of Fig. 7.3, assuming that the inductance of round leads of radius r per unit length is

$$L_0 = (\mu_0/\pi)\ln[(a-r)/r] \tag{a}$$

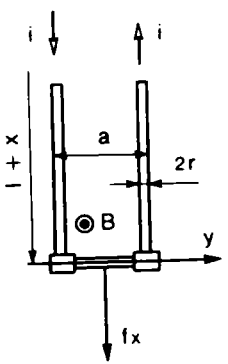


Fig. 7.3

Solution. As the connector shifts a distance x, the inductance $L = L_0(l + x)$ grows, so that

$$\partial L/\partial x = L_0 \tag{b}$$

With the current i being constant, the increment in the electromagnetic energy is

$$dW = i^2 L_0 dx/2 \tag{c}$$

and the energy delivered by the source is given by

$$dW_s = id(Li) = i^2 L_0 dx \tag{d}$$

Hence, by (7.6) we have

$$f_x = \partial W_s/\partial x - \partial W/\partial x = i^2 L_0/2 \tag{e}$$

A joule for a joule. When forces do the work dA in the absence of supply sources, it is evident that the system must expend the same amount of energy specified by (7.7). This conclusion directly follows the law of energy conservation.

But in the examples illustrated above we have pointed out that the field energy grows exactly by the same amount, i.e.

$$dA = dW \tag{7.10}$$

if the system is connected to a power source maintaining a voltage on capacitors (in

considering the work done by the forces of an electrostatic field (Example 7.1) or if the power source maintains a constant current in inductive loops (in considering the work done by the forces of a magnetic field (Example 7.2). In both cases the source furnishes the energy $dW_s = 2dW$. This law was formulated by Kelvin in about 1860.

Equation (7.7) just expresses the law of energy conservation, whereas Eq. (7.10) should sooner cause surprise than appear to represent a realistic situation. The rule (7.10) is a purely general statement, sometimes called a joule-for-joule rule: as the field forces perform the work in 1 joule, the field energy rises by 1 joule. Obviously, the source of voltage must then furnish a double amount of energy, exactly 2 joules. The rule (7.10) was mistaken more than once for the law of conservation of energy. Where this rule applies, the force is expressed as

$$f_\eta = \partial W/\partial \eta \tag{7.11}$$

rather than as given by (7.8).

Without resorting to the general expressions of analytical mechanics, we will only point to a sufficient similarity between the discussed cases, i.e. between (1) the force f_η due to changes in capacitance $C(\eta)$ and (2) the force f_η due to changes in inductance $L(\eta)$.

In the first case where the source of potential is not present, by (7.8) we get

$$f_\eta = -\partial(u^2C)/\partial\eta/2$$

at $Cu = q$ = constant.

Performing the differentiation at the given condition, we find that

$$f_\eta = u^2(\partial C/\partial\eta)/2 \tag{7.12a}$$

i.e. the force is precisely $\partial W/\partial\eta$ at a constant u.

Similarly, for the second case,

$$f_\eta = i^2(\partial L/\partial\eta)/2 \tag{7.12b}$$

i.e. the force is precisely $\partial W/\partial\eta$ at a constant i.

In deriving the force expression, we can reason from various points on whether or not the source supplies energy to the system of interest. For example, the source does not deliver energy to an electrostatic system if the charges on the plates remain constant; as regards an electromagnetic system, the source does not excite the field windings if the flux linkages do not change. If the voltage on the plates of the electrostatic system or the current in the windings of the electromagnetic system remains invariable, then the systems in question receive energy from the sources. Considering the same state of the system at the given moment, the specified forces cannot certainly depend on the probable character of changes of the energy in the system. Thus the attractive force on the plates of a parallel-plate capacitor does not depend on an increase (u remains constant) or decrease in the energy (q remains constant) when bringing the plates closer together.

Forces expressed in terms of field vectors, current, and charge. It is sometimes convenient to define the forces by resorting to the concept of a field but only partially. Thus the force acting on a charge in an electric field is found from the expressions

$$d\mathbf{f} = \mathbf{E}dq, \text{ or } \mathbf{E}\xi dS, \text{ or } \mathbf{E}\rho dV \tag{7.13}$$

and the force on current-carrying wires from the expressions

$$d\mathbf{f} = id\mathbf{l} \times \mathbf{B} \text{ or } \mathbf{J}dV \times \mathbf{B} \tag{7.14}$$

It is of importance to note that here **E** and **B** are the vectors of an external field whose calculation does not include the effect of that charge (dq, ξ, or ρ) or current ($id\mathbf{l}$ or $\mathbf{J}dV$) for which we need to determine the force.

Thus according to (6.47), a cylindrical wire carrying current i experiences a transverse force $f_0 = B_0 i$ per unit length in an external uniform field $\mathbf{B}_0$ regardless of whether it is a copper or iron wire, though in the iron wire the magnetic induction is $B \approx 2B_0$.

Of course, the result that follows from formulas (7.13) and (7.14) cannot and must not in all cases differ from the result deduced from the general energy expressions. For example, the force per unit active area of the plates of a parallel-plate capacitor is

$$f_0 = E\xi/2 \tag{7.15a}$$

which is easy to derive from formula (7.12a), i.e. from energy relationships. The same result follows from (7.13) considering that the plate charge of density $\xi = q/S$ is exposed to a field whose strength is equal to half the strength of the field between the plates, since the other half is due to the field of the charge ξ and hence must not be taken into consideration.

The analysis above permits us to replace (7.15a) for an air capacitor by the expression

$$f_0 = \xi^2/2\varepsilon_0 = \varepsilon_0 E^2/2 \tag{7.15b}$$

since $E = D/\varepsilon_0 = \xi/\varepsilon_0$.

In evaluating the forces by energy equations, there is no need for stipulations of an internal or external field. It should be noted that these stipulations would require more precise explanations.

Example 7.3. Determine the force for the case discussed in Example 7.2 (see Fig. 7.3) using (7.14).

Solution. It is only the field of parallel leads that must be taken into account in determining the force on the crosspiece. This field of leads must be equal to half the field of the same parallel leads but stretching on both sides of the cross section being considered:

$$B = i[1/y + 1/(a - y)]\mu_0/4\pi$$

In this expression the first term specifies the field of the left lead, and the second the field of the right lead. Both components of the field have the same direction (out of the paper).

Using (7.14), we find the sought-for force by integrating between $y = r$ and $y = a - r$:

$$f_x = (\mu_0/4\pi)i^2 \int_{y=r}^{a-r} [1/y + 1/(a - y)]dy = (\mu_0/2\pi)i^2 \ln [(a - r)/r]$$

The calculation based on energy considerations gives the same result [see Example 7.2, formulas (a) and (e)].

On the effect of internal inductance L_i. In calculating the force on the crosspiece (see Fig. 7.3, Example 7.2), we disregarded internal inductance. The role of internal inductance in nonferromagnetic wires is of little importance. As for current-carrying ferromagnetic wires, the field energy in the air space between the wires may

prove to be close to or of the same order of magnitude as the energy of the magnetic field inside the ferromagnet. The force must then be calculated by the formula

$$F = i^2(dL/dx)/2$$

where

$$L = i^2\{(\mu_0/\pi)\ln[(a - r)/r] + L_i\}/2 \tag{7.16}$$

Since this equation defines force, the internal inductance here must obviously be specified proceeding from energy considerations. Fig. 7.4 shows the relation between L_i and the square of current; the values of L_i are found from the measured values of force and current by applying expressions (7.16). The upper curve relates to the circuit supplied with direct current, and the lower curve to the circuit with an ac supply. The internal inductance decays fast with growing current because of magnetic saturation; the inductance is lower in the circuit fed from the ac source since the time-varying magnetic field is not capable of deep penetration into a massive wire (see Sec. 8.3). In Fig. 7.4 is also shown the value of air inductance L_e calculated by the common formula; it does not depend on current.

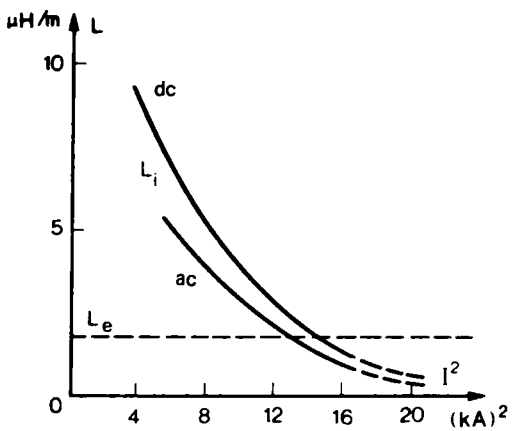


Fig. 7.4

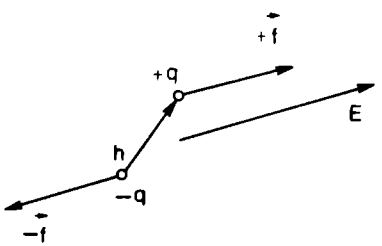


Fig. 7.5

The torque and force experienced by dipoles and polarized bodies. Using the above expressions, we can easily find the torque **T** (moment of the couple of forces) experienced by the dipole (a pair of charges) illustrated in Fig. 7.5. By definition, the mechanical torque is the moment arm **h** times the force $\mathbf{f} = q\mathbf{E}$:

$$\mathbf{T} = \mathbf{h} \times q\mathbf{E} = \mathbf{p} \times \mathbf{E} \tag{7.17}$$

where $\mathbf{p} = q\mathbf{h}$ is the electric dipole moment.

If the field is nonuniform, the forces on the charges $\pm q$ are out of balance. Let the field at the point of location of a negative charge $-q$ be equal to **E** and the field at the point of location of $+q$ be higher than **E** by $h\partial\mathbf{E}/\partial h$ which is a derivative with respect to the direction **h**. The distance h is assumed to be so small that we can restrict ourselves to a linear approximation in considering the change in the field

strength. The force on the entire dipole is then

$$\mathbf{f} = qh\partial\mathbf{E}/\partial h = p\partial\mathbf{E}/\partial h \tag{7.18}$$

The above operation of space differentiation of the vector with respect to the direction $\mathbf{h}$ is expressed in vector analysis in terms of the del operator:

$$h\partial\mathbf{E}/\partial h = (\mathbf{h}\nabla)\mathbf{E}$$

Here the parantheses indicate that we should first take the scalar product of vector $\mathbf{h}$ and del operator ∇ and only then can we perform the differentiation:

$$(\mathbf{h}\nabla)\mathbf{E} \equiv x(\partial E/\partial x) + y(\partial E/\partial y) + z(\partial E/\partial z)$$

In a polarized medium, the electric moment per unit volume is equal to $\mathbf{P}$ and hence the volume force on this medium is

$$\mathbf{f} = (\mathbf{P}\nabla)\mathbf{E} \tag{7.18a}$$

or, on a linear medium where $\mathbf{P} = \varepsilon_0 k\mathbf{E}$,

$$\mathbf{f} = \varepsilon_0 k(\mathbf{E}\nabla)\mathbf{E} \tag{7.18b}$$

where $k = \varepsilon - 1$. The existence of such forces was first revealed by William Thomson (lord Kelvin), so the forces given by (7.18) can by right be called Kelvin forces.

Applying the known transform

$$(\mathbf{E}\nabla)\mathbf{E} = (\nabla\mathbf{E}^2 - \mathbf{E} \times \operatorname{curl}\mathbf{E})/2$$

we find that in a field with nonzero curl,

$$\mathbf{f} = \varepsilon_0 k\nabla\mathbf{E}^2/2 \tag{7.18c}$$

This means that a dielectric is acted upon by a force proportional to the gradient $\mathbf{E}^2$: the dielectric is drawn into the region of a higher field.

Similar formulas are derived for a magnetic field and magnetized bodies. Thus, the force on a magnetic medium is

$$\mathbf{f} = \mu_0(\mathbf{M}\nabla)\mathbf{H} \tag{7.19a}$$

or

$$\mathbf{f} = \mu_0(\mu - 1)(\mathbf{H}\nabla)\mathbf{H} \tag{7.19b}$$

or, when the region in question has curl $\mathbf{H} = 0$, the magnetic medium experiences the force

$$\mathbf{f} = \mu_0(\mu - 1)\nabla\mathbf{H}^2/2 \tag{7.19c}$$

The latter formula tells us that ferromagnetic particles are always drawn into the region of the highest field strength.

For a polarized medium, the torque per unit volume in accordance with (7.17) is given by

$$\mathbf{T}_0 = \mathbf{P} \times \mathbf{E}$$

or

$$\mathbf{T}_0 = \mu_0\mathbf{M} \times \mathbf{H} \tag{7.20}$$

In the case of a homogeneous polarization and a field independent of coordinates,

the total torque within the limits of the body under study (**M** and **H** are constant within V) assumes the form

$$\mathbf{T} = \mu_0 \mathbf{M} \times \mathbf{H} V \tag{7.21}$$

When using formulas for the mechanical interaction between a polarized body and a field, one should consider only the external component of field strength. In an isotropic substance, the vectors of the entire field and polarization are always in the same direction. Therefore, the vector product is different from zero only in bodies showing shape anisotropy, in which case the vector **M** does not coincide in direction with the vector of an external field strength (see Fig. 6.4). All the same here the field strength vector of the entire field is parallel to the vector **M** if a body of anisotropic shape, such as an elongated ellipsoid, is made of an isotropic substance.

Assume an elongated ellipsoid, say, a needle, is placed in an external uniform field $\mathbf{H}_0$ so that its axis is not parallel to the field. The torque it experiences is

$$\mathbf{T} = \mu_0 \mathbf{M} \times \mathbf{H}_0 V$$

If the external field in question has only the xth and zth components, the magnetization terms given by (6.25) take the form

$$M_x = kH_x = kH_{0x}/(1 + kN_x)$$
$$M_z = kH_z = kH_{0z}/(1 + kN_z)$$

where $k = \mu - 1$.

The torque then has only the yth component:

$$T = T_y = \frac{\mu_0 k^2 H_{0x} H_{0z} (N_x - N_z)}{(1 + kN_x)(1 + kN_z)} V \tag{7.22}$$

This formula definitely shows that the torque indeed stems from anisotropy of the form; it is nonzero only when $N_x \neq N_z$. The torque is positive at $N_x > N_z$, which means that it tends to align the needle in the field direction. Reversing the field direction does not affect the torque since in the product $H_{0x}H_{0z}$ each cofactor changes sign.

Let us note in conclusion that the substitution of **H** for $\mathbf{H}_0$ into (7.20) reduces the result to zero:

$$T_0 = \mu_0 k (H_z H_x - H_x H_z) \equiv 0$$

Comparison of two systems of Fig. 7.6. It is interesting to compare the two simple systems depicted in Fig. 7.6*a* and *b*. In each system, the rotor can turn about the axis *0*; each rotor carries one turn of wire (winding) disposed in a pair of diametrically opposite slots. A current i_1 passes through each winding. The stator of the first system represents a coaxial ring. In the second system, the stator has salient poles partially enclosing the rotor. The magnetic fields of both systems can be considered identical within the region corresponding to the angle subtended by the pole pieces (see Fig. 7.6*b*). But the torques exerted on the rotors of the systems being compared are different! Indeed, using energy expression (7.3) and determining the torque by (7.11) and (7.9), we find that

$$T = \partial W/\partial \alpha = (\partial/\partial \alpha)(i_1^2 L_1/2 + i_2^2 L_2/2 + i_1 i_2 M)$$

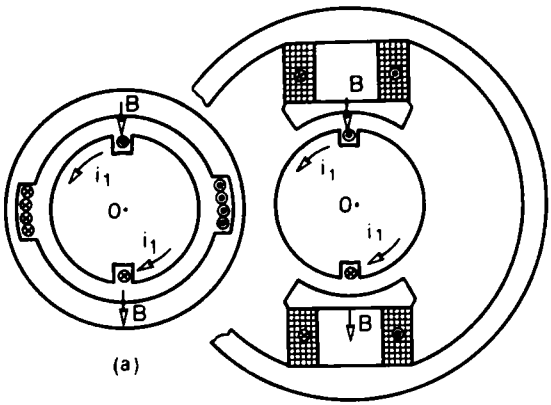


Fig. 7.6

But in the first system (see Fig. 7.6*a*) the rotation of the rotor causes a change only in the mutual inductance, with L_1 remaining constant, whereas in the second system the self-inductance of the rotor winding rises as the rotor is rotating. This is because after rotor turning the magnetic flux is able to complete the magnetic circuit through the salient poles, at least partially. That is why the rotor of the second system tends to turn under the action of the magnetic field of its own current, even in the absence of the current in the stator winding; the rotor strives to take a position in which its self-inductance is at a maximum and this corresponds to its turning by $\pi/2$ with respect to the position shown in Fig. 7.6*b*. In the system of Fig. 7.6*a*, however, the rotor does not experience any torque when $i_2 = 0$ and $i_1 \neq 0$.

Maxwell's rule. The following rule can apply whenever it is necessary to find the forces due to a magnetic field. Current-carrying conductors or other bodies immersed in a magnetic field always have a tendency to motion such that the magnetic flux or total magnetic flux linkage increases.

7.2. Field Energy and Its Flow

Umov's and Poynting's theorems. The analysis of energy transfer in an electromagnetic field can be made by use of Umov's general theorem that corresponds to the law of energy conservation and to the concepts of physical localization of energy: if a change of energy within a volume V takes place, then the flux of energy that flows through the surface bounding V must be equal to this change of energy in the region.

On passing from energy to power, a new formulation of the theorem reads: the rate of energy decrease, $-\partial W/\partial t$, in a volume V is equal to the flow of power going out through the surface bounding this volume. Denoting the density of power flow by vector $\mathbf{Q}$, this theorem takes on such a mathematical form:

$$-\partial W/\partial t = -\int (\partial w/\partial t) dV = \oint \mathbf{Q} d\mathbf{S} \tag{7.23}$$

where w identifies the volume density of energy.

By applying the Ostrogradsky-Gauss theorem to (7.23), we can transform the statement of Umov's theorem into a differential expression:

$$-\partial w/\partial t = \mathbf{div}\,\mathbf{Q} \tag{7.24}$$

N.A. Umov, professor of the Moscow University, presented this theorem in his doctoral dissertation (1874). The theorem relates to the mechanics of continua in the presence of moving bodies, both solid and liquid, or gaseous. British scientists Poynting and Heaviside constructed the proof of an analogous theorem 10 years later, independent of each other and Umov's works. They applied it to the energy of an electromagnetic field in the absence of moving bodies in the region under examination. The derivation of the equation for a power density vector in an electromagnetic field also refers to the same period of time. The equation has the form

$$\mathbf{Q} = \mathbf{\Pi} = \mathbf{E} \times \mathbf{H} \tag{7.25}$$

It is valid for the region being considered in the absence of moving bodies and any internal sources of energy. The vector $\mathbf{\Pi}$ is known as the Poynting vector; it can also be written in the form other than that of (7.25) by adding the terms whose divergence is equal to zero and hence does not disturb the condition (7.24).

Derivation of Poynting's theorem. Let us refer to Maxwell's equations

$$\begin{aligned} \text{curl}\,\mathbf{H} &= \sigma\mathbf{E} + \partial\mathbf{D}/\partial t \\ \text{curl}\,\mathbf{E} &= -\partial\mathbf{B}/\partial t \end{aligned} \tag{7.26}$$

It is assumed here that internal sources of energy and moving bodies are not present[1]. What can serve as a key to the analysis of these equations from the viewpoint of energy transformations is Joule's law in differential form

$$P_0 = \mathbf{JE} = \sigma E^2 \tag{7.27}$$

where P_0 is the power per unit volume.

Consider a field in which only the dissipation of electromagnetic energy, σE^2, takes place and other kinds of energy conversion are nonexistant; the assumption also is that both the electric and the magnetic field must not certainly vary with time. In accordance with the adopted ideas of energy, we can reason thus: if the power dissipated in an element of volume is $\sigma E^2/dV$, then the same flow of power crosses the unit area on the surface bounding the volume V:

$$-\oint \mathbf{\Pi} d\mathbf{S} = \int \sigma E^2 dV \tag{7.28}$$

Here $\mathbf{\Pi}$ is the rate of flow of energy per unit area; the minus sign on the left side shows that the normal $d\mathbf{S}$ is directed outward and hence the integral (without the minus sign) represents the outward flow of energy. Expression (7.28) can be

[1] In the presence of bodies moving in a magnetic field, Eqs. (7.26) are not valid even at low velocities. One of the causes, as revealed by Maxwell, is that the current density assumes the form

$$\mathbf{J} = \sigma(\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

and so its representation by a simple product $\sigma\mathbf{E}$ is impossible.

written in differential form:

$$-\operatorname{div}\boldsymbol{\Pi} = \sigma E^2 = \mathbf{JE} \tag{7.29}$$

This means that the rate of decrease of the Poynting flux is equal to the rate of electromagnetic energy dissipated in heat according to Joule's law.

Return now to the general field equations. To introduce the term $\sigma \mathrm{E}^2$ into them, multiply the first equation by $\mathbf{E}$ to obtain

$$\mathbf{E}\operatorname{curl}\mathbf{H} = \sigma E^2 + \mathbf{E}\partial\mathbf{D}/\partial t \tag{7.30}$$

To represent the left side of this equality (with σE^2 on its right side) as divergence of a certain vector, subtract from formula (7.30) Maxwell's second equation preliminarily multiplied by $\mathbf{H}$. Applying the vector formula

$$\mathbf{A}\operatorname{curl}\mathbf{B} - \mathbf{B}\operatorname{curl}\mathbf{A} = -\operatorname{div}(\mathbf{A}\times\mathbf{B})$$

to Maxwell's equations, we get

$$\mathbf{E}\operatorname{curl}\mathbf{H} - \mathbf{H}\operatorname{curl}\mathbf{E} = -\operatorname{div}(\mathbf{E}\times\mathbf{H}) = \sigma\mathbf{E}^2 + \mathbf{E}\partial\mathbf{D}/\partial t + \mathbf{H}\partial\mathbf{B}/\partial t \tag{7.31}$$

But in the conditions of a static field, when $\partial/\partial t = 0$, the expression takes the form

$$-\operatorname{div}(\mathbf{E}\times\mathbf{H}) = \sigma E^2 \tag{7.32}$$

Its comparison with (7.29) allows us to express the electromagnetic power density flow in the form

$$\boldsymbol{\Pi} = \mathbf{E}\times\mathbf{H} \tag{7.33}$$

i.e. as the Poynting vector.

Applying the same interpretation of vector $\boldsymbol{\Pi} = \mathbf{E}\times\mathbf{H}$ to a time-varying field, it is easy to understand the meaning of all the terms on the right of identity (7.31): the term $\mathbf{E}\partial\mathbf{D}/\partial t$ is the power expended in creating an electric field

$$\mathbf{E}\partial\mathbf{D}/\partial t = \partial w_e/\partial t \tag{7.34}$$

and the term $\mathbf{H}\partial\mathbf{B}/\partial t$ is the power needed to produce a magnetic field

$$\mathbf{H}\partial\mathbf{B}/\partial t = \partial w_m/\partial t \tag{7.35}$$

Indeed, visualize a capacitor with an ideal dielectric whose conductivity is zero. Assume the capacitor charges at a constant current (i = constant at du_C/dt = constant). Then, $\sigma\mathbf{E} = 0$ and $\mathbf{H}\partial\mathbf{B}/\partial t = 0$ (since i = constant). Hence,

$$-\operatorname{div}\boldsymbol{\Pi} = -\operatorname{div}(\mathbf{E}\times\mathbf{H}) = \mathbf{E}\partial\mathbf{D}/\partial t \tag{7.36}$$

The inflow of energy per unit time dt is only spent on producing the electric field in the dielectric. This energy referred to a unit volume is

$$dw_e = \mathbf{E}d\mathbf{D} \tag{7.37}$$

For a linear dielectric, in which $\mathbf{D} = \varepsilon\varepsilon_0\mathbf{E}$,

$$dw_e = \varepsilon\varepsilon_0 d\,E^2/2 \tag{7.38}$$

and, hence, we may regard

$$w_e = \varepsilon\varepsilon_0 E^2/2 \tag{7.39}$$

as the energy density of the electric field.

For a nonlinear dielectric, the power $\mathbf{E}\partial\mathbf{D}/\partial t$ must be thought of as the power expended in creating the field, though not necessarily all this power contributes to an increase in the field energy. For a dielectric in which the **D-E** relation takes the form of a loop, the energy delivered to a growing field (D varies from $-D_{\max}$ to $+D_{\max}$) is higher than the energy given up by the field as it changes from $+D_{\max}$ to $-D_{\max}$. The sum of these energies

$$\oint \mathbf{E}d\mathbf{D} = \int_{-D_{\max}}^{+D_{\max}} \mathbf{E}d\mathbf{D} + \int_{+D_{\max}}^{-D_{\max}} \mathbf{E}d\mathbf{D} = w_h \tag{7.40}$$

is the energy converted to heat due to hysteresis (the energy lost in hysteresis during each hysteresis cycle). As in the text above, the energy referred to here is certainly the energy per unit volume. It should also be kept in mind that the electric flux density is assumed to rise and fall monotonically.

Similar reasonings permit us to conclude that the summand $\mathbf{H}\partial\mathbf{B}/\partial t$ in (7.31) expresses the power expended in establishing the magnetic field in a unit volume. At μ being constant (for a linear magnetic medium), the volume energy density for a magnetic field is given by

$$w_m = \mu\mu_0 H^2/2 \tag{7.41}$$

For a nonlinear medium, the energy expended in producing the magnetic field in a unit volume is

$$dw_m = \mathbf{H}d\mathbf{B} \tag{7.42}$$

This equality and (7.37) are similar in meaning.

The loss due to magnetization reversal, in particular a hysteresis loss, is expressed by the formula similar to (7.40):

$$\oint \mathbf{H}d\mathbf{B} = w_h \tag{7.43}$$

This equality is known as Warburg's law.

Rewriting expression (7.31) in integral form

$$\begin{aligned} -\oint (\mathbf{E}\times\mathbf{H})d\mathbf{S} = -\oint \Pi d\mathbf{S} &= \int \sigma E^2 dV + \int \mathbf{E}(\partial\mathbf{D}/\partial t)dV + \int \mathbf{H}(\partial\mathbf{B}/\partial t)dV \\ &= \partial W_\theta/\partial t + \partial W_e/\partial t + \partial W_m/\partial t \end{aligned} \tag{7.44}$$

we come to the most popular formulation of Poynting's theorem which reads: the sum of the rate at which the energy of the electromagnetic field increases inside a volume V ($\partial W/\partial t = \partial W_e/\partial t + \partial W_m/\partial t$) and the rate at which an amount of the electromagnetic field energy inside this volume converts to other kinds of energy ($\partial W_\theta/\partial t$) is equal to the flow of power of the electromagnetic field going in through the closed surface S bounding this volume; this power flow is equal to the vector flux $\Pi = \mathbf{E}\times\mathbf{H}$.

For a simple linear medium characterized by the parameters σ, ε, and μ (each being independent of the field strength), the rate of rise in the field energy is

$$\int \varepsilon\varepsilon_0 \mathbf{E}(\partial\mathbf{E}/\partial t)dV + \int \mu\mu_0 \mathbf{H}(\partial\mathbf{H}/\partial t)dV = \partial W/\partial t$$

and the rate at which the electric energy changes to heat is

$$\partial W_\theta/\partial t = \int \sigma E^2 dV$$

For a linear circuit with lumped parameters L, C, and r, the expressions have the form

$$\partial W/\partial t = (\partial/\partial t)\sum (Li^2 + Cu_C^2)/2$$

and

$$\partial W_\theta/\partial t = \sum i^2 r$$

Two-wire transmission line. From the viewpoint of the theory presented above, the power in a two-wire line is transmitted via the isolating medium surrounding the conductors, where the vectors of the electric and the magnetic field are mutually orthogonal and the Poynting vector is parallel to the conductors and is in the direction of energy transfer. The conductors here play only the role of guides. Fig. 7.7*a* illustrates schematically the field in a transverse cross section. The dashed lines represent the magnetic field lines. At each point, the vectors of field strength are tangent to respective lines. The direction of Poynting's vector is indicated by the cross symbols (tails of the arrows) denoting that the vector is into the page.

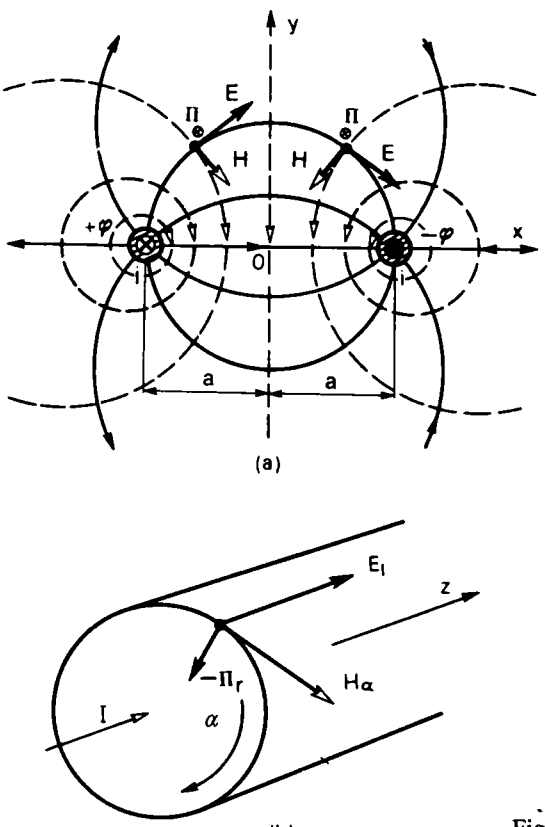


Fig. 7.7

Inside the conductor (Fig. 7.7*b*), there is only an axial component of the electric field strength (charges move along the wire), which is equal to the current density divided by the conductivity, $E_l = J/\sigma$. Such an axial component certainly exists on the external surface of the conductor as well; it is commonly very small as compared to the normal component of the electric field strength, E_n. For example, a high-voltage power line shows such values of the components: $E_l \approx 1.8$ V/m and $E_n \approx$ $\approx$ 400 kV/m.

But the axial component of an electric field determines a Poynting vector component directed along the radius toward the conductor axis (see Fig. 7.7*b*). The flow of power entering the wire is due to Joule's losses inherent in the conductor.

Poynting vector expression in modified form. By substituting the electric field strength $\mathbf{E} = -\nabla\varphi - \partial\mathbf{A}/\partial t$ into the Poynting vector expression, we obtain

$$\mathbf{\Pi} = \mathbf{E} \times \mathbf{H} = -\nabla\varphi \times \mathbf{H} - (\partial\mathbf{A}/\partial t) \times \mathbf{H}$$

Using the known vector formula

$$\text{curl}\,(\varphi\mathbf{F}) = \varphi\nabla \times \mathbf{F} - \mathbf{F} \times \nabla\varphi$$

we can write the first term on the right of the expression in the form

$$-\nabla\varphi \times \mathbf{H} = \varphi\text{curl}\mathbf{H} - \text{curl}(\varphi\mathbf{H})$$

The term curl ($\varphi\mathbf{H}$) has no divergence, and so it can be omitted for sure in the calculation of power flow; in any case, it vanishes when under consideration is precisely the divergence of vector $\mathbf{\Pi}$.

The power density can thus be written in the form

$$\mathbf{\Pi} = \mathbf{\Pi}_1 - (\partial\mathbf{A}/\partial t) \times \mathbf{H} \tag{7.45}$$

where

$$\mathbf{\Pi}_1 = \varphi\text{curl}\,\mathbf{H} = \varphi\mathbf{J} \tag{7.46}$$

The second term in (7.45) is equal to zero for a static field and so the entire power flow is expressed by (7.46). Expression (7.46) was suggested by J. Slepian (USA) (See also C. O. Hines, *Canad. J. Phys.*, 1952).

It is of interest to apply (7.45) to the case of the two-wire overhead power line considered above. Since A is equal to A_z and $\mathbf{H}$ lies in the plane xy, as shown in Fig. 7.7*a*, the second component in (7.45) is always normal to the wire axis and contributes to the transfer of field energy only in the direction normal to the power line axis (this component represents a reactive power). The flow of energy along the line is expressed only by the first component in (7.45), i.e. by the vector (7.46); inside the conductors, this vector is parallel to the direction of the line axis.

Electromagnetic energy of a wave. As mentioned earlier, Maxwell showed that the solution to the system of equations for the field in a dielectric can give a propagating wave with mutually orthogonal vectors $\mathbf{H}$ and $\mathbf{E}$ lying in the plane perpendicular to the direction of wave propagation; the wave travels along the Poynting vector direction (see Figs. 7.7 and 2.5). The Poynting vector represents the amount of energy carried by the wave per unit time through unit area of the surface normal to the wave direction. Proceeding from this definition, we should suppose that the density of an electromagnetic wave pulse (momentum) is the vector given by

$$\mathbf{g} = \Pi/c^2$$

and that the electromagnetic wave has a mass whose space density is

$$\rho = \Pi/c^3$$

Experiments fully confirm the validity of these concepts. Thus, using the momentum density expression, we can calculate the pressure of electromagnetic waves in agreement with the experimental results.

Complex Poynting's vector theorem. For a sinusoidally varying field and a linear medium, we can represent Maxwell's equation in complex notation. For the first equation to include Joule's losses

$$P_0 = \sigma E^2 = \sigma \dot{\mathbf{E}} \overset{*}{\dot{\mathbf{E}}} \tag{7.47}$$

the right-hand side should contain the product of complex number $\dot{\mathbf{E}}$ and complex number $\overset{*}{\dot{\mathbf{E}}}$ conjugate to the first number. Here $\dot{\mathbf{E}}$ is the complex quantity of the effective field, so that the instantaneous field vector is

$$\mathbf{E}(t) = \sqrt{2}\,\mathrm{Re}\dot{\mathbf{E}}e^{j\omega t}$$

In order to include (7.47), we can just multiply both sides of Maxwell's first equation (3.55) by the complex quantity $\overset{*}{\dot{\mathbf{E}}}$, preliminarily replacing all the terms of the second factor curl $\dot{\mathbf{H}}$ by their conjugates:

$$\overset{*}{\dot{\mathbf{E}}}\,\mathrm{curl}\,\overset{*}{\dot{\mathbf{H}}} = \overset{*}{\dot{\mathbf{E}}}(\sigma \overset{*}{\mathbf{E}} - j\omega\varepsilon_0 \underline{\varepsilon}\, \overset{**}{\mathbf{E}}) \tag{7.48a}$$

Maxwell's second equation (2.14), which includes the complex permeability given by (3.117), i.e.

$$\mathrm{curl}\,\dot{\mathbf{E}} = -j\omega\underline{\mu}\mu_0\dot{\mathbf{H}}$$

should be multiplied by the complex conjugate of magnetic field strength:

$$\overset{*}{\dot{\mathbf{H}}}\,\mathrm{curl}\,\dot{\mathbf{E}} = -\mathrm{j}\omega\underline{\mu}\,\mu_0\mathbf{H}^2 \tag{7.48b}$$

because only in this case can we encounter identical vectors $\dot{\mathbf{E}}$ and $\overset{*}{\dot{\mathbf{H}}}$ in the difference being transformed:

$$\dot{\mathbf{E}}\,\mathrm{curl}\,\overset{*}{\dot{\mathbf{H}}} - \overset{*}{\dot{\mathbf{H}}}\,\mathrm{curl}\,\dot{\mathbf{E}} = -\mathrm{div}(\dot{\mathbf{E}} \times \overset{*}{\dot{\mathbf{H}}}) = \sigma E^2 - j\omega\varepsilon_0\underline{\varepsilon}\,E^2 + j\omega\mu_0\underline{\mu}\,H^2$$
$$= (\sigma + \omega\varepsilon_0\varepsilon'')E^2 + \omega\mu_0\mu\,H^2 + j\omega(-\varepsilon_0\varepsilon\,E^2 + \mu_0\mu\,H^2) \tag{7.49}$$

This transform is similar to (7.31), and the vector $\dot{\mathbf{E}} \times \overset{*}{\dot{\mathbf{H}}}$ is the complex expression of the Poynting vector $\underline{\Pi}$. But now, among the real terms entering into the right side of (7.49), there are other terms, along with Joule's heat, σE^2, that represent the irreversible dissipation of electromagnetic energy in a dielectric (dielectric loss $\omega\varepsilon_0\varepsilon'' E^2$) and in a ferromagnetic (magnetic loss $\omega\mu_0\mu'' H^2$). These three terms add up in a simple way: their sum is equal to the mean power dissipated for one period and referred to a unit volume, P_0.

The remaining imaginary part of (7.49) which is

$$j\omega(-\varepsilon_0\varepsilon' E^2 + \mu_0\mu' H^2) = jQ_0 \tag{7.50}$$

represents the reactive power consumed in a varying field and referred to a unit volume. The reactive power includes a magnetic component $\omega\mu_0\mu' H^2$ positive in

sign and an electric component $-\omega\varepsilon_0\varepsilon' E^2$ which is negative. The signs of these components are purely conditional. According to the definition agreed upon in Europe, the consumed reactive power is positive for an inductive reactance, $Q = Q_L = I^2\omega L > 0$, and negative for a capacitive reactance, $Q = -Q_C = -I^2/\omega C = -U_C^2\omega C < < 0$. The formula of complex power should then contain the conjugate of current I, i.e. $\underline{S} = \dot{U}\dot{I}$, and the Poynting vector expression should have the conjugate of magnetic field strength, $\mathbf{H}$.

The rule adopted in the USA is other. By this rule, $\underline{S} = \dot{U}\overset{*}{\dot{I}}$ and $\underline{\Pi} = \dot{\mathbf{E}} \times \overset{*}{\dot{\mathbf{H}}}$, and only imaginary components determining the reactive power change sign.

The statement of (7.49) for the total complex power flowing into region V bounded by the closed surface S has the form

$$\begin{aligned}\underline{S} = P + jQ &= -\oint \underline{\Pi} d\mathbf{S} = -\oint (\dot{\mathbf{E}} \times \overset{*}{\dot{\mathbf{H}}}) d\mathbf{S} \\ &= \int (\sigma E^2 + \omega\varepsilon_0\varepsilon'' E^2 + \omega\mu_0\mu'' H^2) dV \\ &\quad + j\omega \int (\mu_0\mu' H^2 - \varepsilon_0\varepsilon' E^2) dV \end{aligned} \tag{7.51}$$

This equation includes the most general expression of reactive power and also reflects the law of conservation of this power. It is now evident from this equation that the phases of both an electric and a magnetic field strength play no role in the algebraic summation of reactive components.

As for an alternating current, the representation of the Poynting vector in a form similar to that of (7.45) is also possible:

$$\underline{\Pi} = \dot{\varphi}\overset{*}{\mathbf{J}} - j\omega\dot{\mathbf{A}} \times \overset{*}{\dot{\mathbf{H}}} \tag{7.52}$$

Energy flow in magnetic and dielectric media. The analysis of the Poynting vector permits us to derive expressions for magnetic and dielectric losses and understand the essence of reactive power components in terms of the pulsation of field energy. Consider a ferrite cylinder (Fig. 7.8) placed in a transverse field, $\dot{H}_0$. The ferrite conductivity is so small that the magnetic field due to induced currents can be

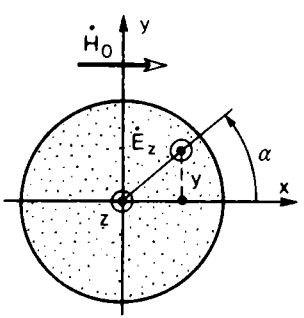


Fig. 7.8

neglected. In this case, we may directly apply formula (5.30) to a varying field too, replacing of course $\mathbf{E}$ by $\mathbf{H}$ and ε by μ. Assuming that the medium surrounding the cylinder has a permeability equal to 1, we find that the field inside the cylinder is uniform:

$$\dot{H} = \dot{H}_x = 2\dot{H}_0/(\mu + 1) \tag{7.53}$$

The axial component of the induced electric field (see Fig. 7.8)

$$\dot{E} = \dot{E}_z = -j\omega\dot{\Phi} \tag{7.54}$$

is equal here to the induced emf per unit length of the loop enclosing the magnetic flux in question, $\dot{\Phi} = \underline{\mu}\mu_0 \dot{H}y$.

Since $\dot{E} = \dot{E}_z$ and $\dot{H} = \dot{H}_x$, the only component of the Poynting vector is in the direction of the y axis:

$$\underline{\Pi} = \underline{\Pi}_y = \dot{E}_z H_x^* = -j\omega\underline{\mu}\mu_0 H^2 y \tag{7.55}$$

The absorption of this vector flux per unit volume, i.e. $-\operatorname{div}\underline{\Pi}$, is found quite readily:

$$-\operatorname{div}\underline{\Pi} = -\partial\underline{\Pi}_y/\partial y = j\omega\underline{\mu}\mu_0 H^2 = j\omega(\mu' - j\mu'')\mu_0 H^2 \tag{7.56}$$

The real part of $-\operatorname{div}\underline{\Pi}$ is the active component of consumed power related to a unit volume

$$\operatorname{Re}(-\operatorname{div}\underline{\Pi}) = \omega\mu''\mu_0 H^2 = P_0 \tag{7.57}$$

The imaginary part of $-\operatorname{div}\underline{\Pi}$ is the consumed reactive power Q_0 also related to a unit volume. The reactive power (consumed) is positive in accordance with the adopted rule. If we were to solve a similar problem for a dielectric cylinder immersed in a transverse electric field, we would find that the consumed reactive power is negative.

7.3. Energy Transfer in Electrical Machines

When a mover drives the rotor of an electrical machine and the electromagnetic energy generated in the stator flows into a transmission line, the entire flux of energy converted in the generator passes from rotor to stator through the air gap as the Poynting flux. This is exactly the way of energy transfer in a synchronous machine (Fig. 7.9*a*). The vector flux passes through a coaxial cylindrical surface enclosing the rotor and lying in the air gap between the stator and rotor. The power P_0 transferred per unit length of the machine is expressed by the Poynting flux:

$$P_0 = \oint \boldsymbol{\Pi} d\mathbf{S} = \oint \Pi_r dS \tag{7.58}$$

Here the scalar product includes only the radial component of the Poynting vector. This component is expressed in terms of the zth component of **E** and the tangential (αth) component of **H**:

$$\Pi_r = -E_z H_\alpha \tag{7.59}$$

But as shown in (6.45), the axial component of the electric field induced by the varying magnetic field at $r_1 \leqslant r \leqslant r_2$ is

$$E = E_z = \omega r \mu_0 H_r$$

The positive reference directions for α and z and also the positive direction of ω are interrelated in the same way as in the normal right-handed coordinate system. Therefore,

$$\Pi_r = -\omega r \mu_0 H_r H_\alpha \tag{7.60}$$

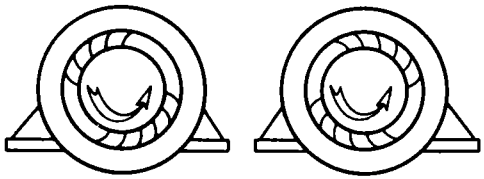


Fig. 7.9 (a) (b)

For the field of Fig. 7.9*a*, the sign of the product of three cofactors, $\omega H_r H_\alpha$, is negative, i.e. the power flows from rotor to stator[1].

It is different with the example illustrated in Fig. 7.9*b*: the power flows from stator to rotor, $\Pi_r < 0$. This is now the case of an electric motor.

In the first case of Fig. 7.9*a* the energy flows from rotor to stator; **H** field lines make an angle with the radius, which is in agreement with Faraday's visualization of magnetic line tensioning: the tensioning of magnetic lines tends to slow the rotor down, and the turbine or any other driver opposes this tensioning and performs the mechanical work. This work causes the Poynting flux of energy to flow to the stator and then to a power line.

In the second case (see Fig. 7.9*b*), the rotating magnetic field of the stator tends to pull along and turn the armature. This is a motor. The energy now flows from the power line to the stator and then, as the Poynting flux, to the armature to do the mechanical work. The flow of power here is toward the axis, and so the Poynting vector has a negative radial component.

Example 7.4. Find the Poynting flux through an imaginary cylindrical surface of radius $r_1 \leqslant r \leqslant r_2$ that envelopes the rotor of a synchronous machine (see Figs. 6.9 and 6.11*a*), using formulas (6.40) and (6.41) that describe the field distribution in the air gap of the machine of Fig. 6.9. It is exactly this pattern of field distribution that exists in the air gap of a synchronous machine whose armature is driven by a turbine with a speed ω, the stator winding being supplied with alternating current that produces a rotating (traveling) field. To a sufficient degree of approximation the surface current density in three three-phase windings may be taken equal to

$$K_{st} = -H_{\beta 2} = +K_2 \cos\beta = +K_2 \cos(\alpha - \omega t) \tag{a}$$

Solution. The radial component of the Poynting vector that represents the sought-for flux is expressed in terms of the zth component of **E** and the tangential component of **H**.

Assuming that the radius of the imaginary cylinder is equal to r and using formulas (7.60), (6.40), and (6.41), we obtain

$$\Pi_r = -\omega r \mu_0 H_r H_\beta = L\cos^2\beta + M\cos\beta \sin\beta + N\sin^2\beta \tag{b}$$

where the constants L, M, and N can easily be found after appropriate substitution of (6.40) and (6.41). On integrating over the entire cylinder surface,

$$P_0 = \int \Pi_r dS = \int_{\alpha=0}^{2\pi} \Pi_r r d\alpha \tag{c}$$

---

[1] The signs of all cofactors ω, H_r, and H_α are interrelated. Thus, for the case of Fig. 7.9*a*, when $\omega > 0$ the quantity H_r is positive and hence H_α is negative. If $H_r < 0$, then $H_\alpha > 0$, and therefore $\Pi_r = \omega r \mu_0 |H_r H_\alpha|$.

we find that

$$P_0 = (L + N)r\pi \tag{d}$$

because

$$\int_0^{2\pi} \sin x \cos x \, dx \equiv 0$$

Performing the requisite algebraic manipulations needed to determine L and N, we see that quite a few terms cancel and the result obtained does not depend on the chosen value of radius r on condition, of course, that it lies in the interval $r_1 \leqslant r \leqslant r_2$. The final result is of the form

$$P_0 = 2\pi\mu_0\omega K_1'' K_2 r_1^2 r_2^2/(r_2^2 - r_1^2) \tag{e}$$

Discussion of the found solution. The existence and sign of the flow of power are determined by the existence and sign of the angular (space) shift between the surface currents in the stator and rotor. At $K_1 \neq 0$ and $K_2 \neq 0$, the field in the air gap is present, but at $K'' = 0$ the tangential component of the magnetic field strength does not exist, hence nor does the radial component of the Poynting vector. If curent densities K_1'' differ in sign, tangential components of different signs appear (see Fig. 7.9*a* and *b*).

In the presence of moving bodies, it is necessary to take into account the mechanical work being done along with the flow of electromagnetic energy. The work can be expressed by Umov's flux.

The machines in which the rotor produces an electromagnetic energy and the stator carries a dc winding (or just a permanent magnet) do not certainly offer any transfer of energy from stator to rotor, so that the Poynting vector through the air gap is absent. The Poynting vector appears in the rotor itself and its flux goes from the rotor to the power line connected via the slip-ring brushes to the rotor winding where the emf is induced.

Determining the torque T from the analysis of the Poynting flux expression. For the case of the electrical machine considered above, the power P transported as the Poynting flux to the rotor can be expressed by the product of torque T and angular velocity ω. Therefore,

$$T = P/\omega = -\int (\Pi_r/\omega) dS \tag{7.61}$$

or, considering (7.60),

$$T_0 = \int_{\alpha=0}^{\alpha=2\pi} r^2 \mu_0 H_r H_\alpha \, d\alpha \tag{7.62}$$

Expression (7.62) deserves particular notice as it permits evaluating T only from the values of H in the air gap. This formula is also applicable to stationary internal and external cylinders if the field between them obeys formulas (6.40) and (6.41).

A similar method of determining the torque (or some other generalized force with which one body acts on the other) from the electromagnetic field in the gap between any two bodies can also be adapted for other systems. A further analysis of the method leads to the Maxwell-Faraday theory of stress tensors in an electromagnetic field.

7.4. Energy Converters

Along with the vector flux of electromagnetic energy, the flow of other kinds of energy should often be taken into considerations. The vector that represents the flow of other types of energy per unit time has its specific features. In Fig. 7.10 is shown schematically the energy converter where various flows of energy W enter and leave it (subscript 1 denotes energy inflow and subscript 2 energy outflow). As illustrated in Fig. 7.10, the following relations between the incoming and outgoing energy fluxes are possible: the inflow of electromagnetic energy, W_{1em} is smaller than the outflow of the same kind of energy, W_{2em}. This increase of the energy is due to the incoming high-frequency acoustic vibration energy W_{1a}, a significant amount of which goes into raising the electromagnetic field as a result of magnetostriction or electrostriction. The ouflow of acoustic energy, W_{2a}, is therefore smaller than its inflow.

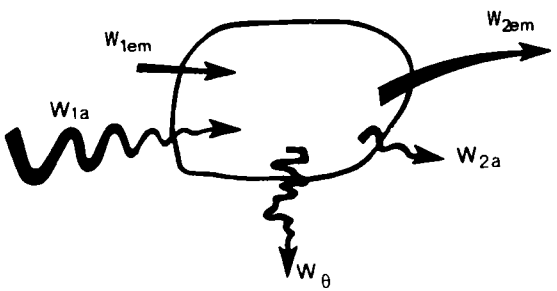


Fig. 7.10

In summary, the inward energy flow W_{1a} can be regarded as the energy of a local source (pumped energy), W_{1em} as the energy of an incoming signal, and W_{2em} as the energy of an amplified signal at the output of the converter. Both flows W_1 can be electromagnetic, but of different frequencies. In the absence of an input signal, no signal at the output must exist. The energy flow W_θ that accompanies any energy conversion is the energy dissipated in heat.

This type of energy transformation often shows a sharply pronounced resonance character — transformation occurs only in a very narrow frequency range specific to both the pumping energy and the signal. The oscillatory interaction often stems from the quantum-molecular properties of a substance, so that the described type of converter is referred to as a quantum oscillator. The physics of processes in such an oscillator is beyond the scope of classical electrodynamics. This principle of interaction underlies the operation of lasers in which the electromagnetic modes can span the range of radio and light frequencies.

7.5. Lifting Force of a Magnet

Figure 7.11 shows schematically two magnets: the first (Fig. 7.11*a*) has a magnetic circuit ending on an armature, and the second (Fig. 7.11*b*) is the electromagnet consisting of two identical halves. If the permeability of the armature is high, the magnet

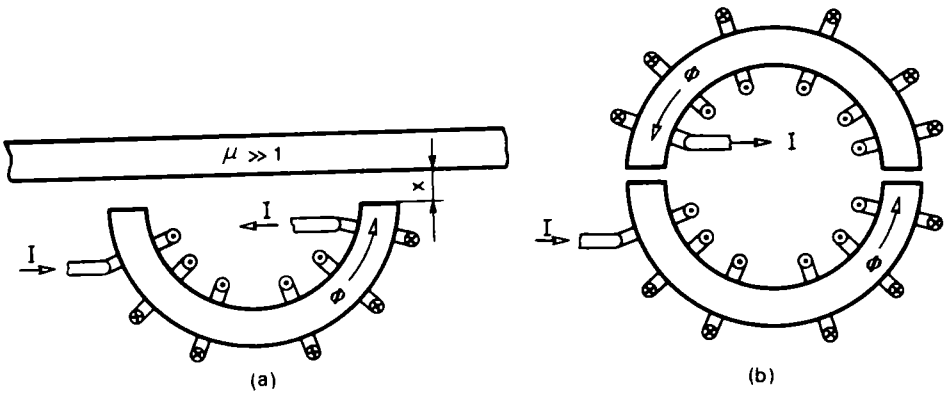


Fig. 7.11

poles and their mirror images in the armature represent two identical systems. The lifting force of such an electromagnet can be very great. An American engineer Joseph Henry constructed an electromagnet with a lifting capacity of a few tons back in the 20s of the last century. Henry took such a great interest in the problem of magnetic pull that he postponed publication of his work on electromagnetic induction for many years. His work on self-induction appeared in 1832. It was Faraday who first described the phenomenon of electromagnetic induction (1831).

The easiest way to determine the lifting force is to apply the principle of energy balance. Assuming the length of the air gap, x, is small as against the dimensions of poles, i.e. taking the field in the gap to be practically uniform, the field energy in the air gap becomes

$$W_0 = V_0 w_0 = 2SxB^2/2\mu_0 \tag{7.63}$$

where $V_0 = 2Sx$ is the air gap volume between the armature and both pole faces of the electromagnet of Fig. 7.11*a*.

Assume the current in the coil is so adjusted in response to small displacements of the armature (to small changes in x) that the magnetic flux $\Phi = BS$ is kept constant. Assume also that the system receives no energy from the source (i.e. $u = d\Phi/dt = 0$), and its energy can change only as a result of changes in the air gap volume. The force acting on the unit area of each pole face then becomes

$$F_0 = (dW_0/dx)/2S = B^2/2\mu_0 \tag{7.64a}$$

or, more accurately,

$$F_0 = MB/2 \approx B^2/2\mu_0 \tag{7.64b}$$

For the poles with a face area of $2S = 0.08\ \text{m}^2$, we find that at $B = 1.5$ T the lifting force is $F = 2SF_0 = 7.16 \times 10^4$ N. Such a magnet can support (lift up) a weight of 7.3 t since 1 N = 0.102 kgf.

The magnetic pull of electromagnets can be very high. An illustrative example can be magnetic suspension of freight cars on support rails that serve as an armature. Electromagnets can rigidly be connected to the car axles and located under the ar-

mature just as in Fig. 7.11*a*. Being attracted to the armature, the electromagnet raises the car axles and thus lifts up the car. Cutting off the current in the electromagnet coil enables lowering the car wheels on the track. This type of suspension lacks completely stable equilibrium, whatever the size of a finite air gap: on the one hand, as it starts descending under gravity, the magnet breaks away from the armature; on the other hand, as it begins to lift up, the magnet sticks to the armature. The reason is that the pull grows with decreasing gap because the reluctance of the magnetic circuit decreases and the induction rises. The use of quick-acting automatic devices eliminates the shortcoming in the characteristic of a magnet: as the gap grows in excess of a certain preset value x_0, the current in the coil rises fast and the electromagnet starts moving up closer to the armature; on the contrary, as the gap becomes smaller than the specified value, the current automatically drops off causing a decrease in the force of attraction. As a result, the freight car remains in the steady suspended state. A remarkable thing is that in magnetic suspension the friction of rest is totally nonexistant and the friction in motion is very negligible, so that even a child can move the freight car and wheel it.

The lifting force of a permanent magnet can approximately be found from formula (7.64).

7.6. Determining Forces from Reluctance

The energy of a magnetic field for any linear portion (μ = constant) of a system can be expressed in terms of the magnetic flux and magnetic potential difference (mpd)

$$W_m = \int \mathbf{HB} dV/2 = \Phi U_m/2 \tag{7.65}$$

where the difference of magnetic potentials between equipotential surfaces *1* and *2* of magnet poles is $U_m = \int_1^2 \mathbf{H} d\mathbf{l}$ and the magnetic flux is $\Phi = \int \mathbf{B} d\mathbf{S}$.

If an electromagnet or a permanent magnet represents a definite magnetic circuit, it makes sense to use the notions of reluctance and permeance

$$r_m = U_m/\Phi \text{ and } G_m = \Phi/U_m \tag{7.66}$$

and express the energy by the formula

$$W = \Phi^2 r_m/2 = U_m^2 G_m/2 \tag{7.67}$$

When determining the ηth component of force (with i kept constant) from changes in energy, $f_\eta = -\partial W/\partial \eta$, it is convenient in a number of cases to calculate the derivative from reluctance or permeance. The energy of a complex system should best be evaluated from (7.67) as a sum of the energies of individual circuit portions. In the calculation of a simple circuit, it is expedient to express the magnetic flux in terms of the magnetomotive force Θ (or F):

$$\Phi = \Theta/r_m \tag{7.68}$$

In Fig. 7.12 is shown the electromagnetic suspension system of a gyroscope, which is calculated by the methods based on the concepts of a magnetic circuit. In the system of Fig. 7.12, as the gyrowheel starts to spin, the magnetic resistance that opposes the fluxes $\Phi_1, \Phi_2, \ldots$, changes and so does the wheel potential φ_0 relative to the

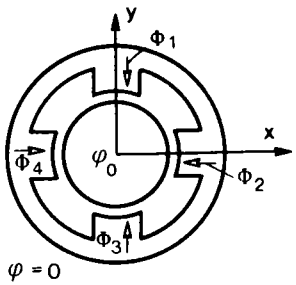


Fig. 7.12

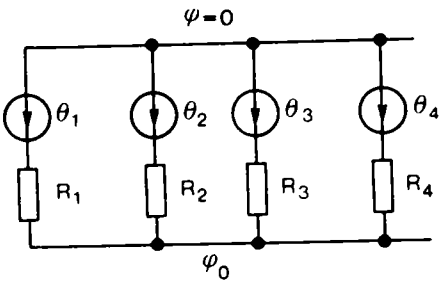


Fig. 7.13

yoke potential conditionally taken equal to zero. The magnetic potential φ_0 can readily be estimated by applying the known calculation methods used in electric circuit theory to an appropriate analog such as that illustrated in Fig. 7.13. The magnetic potential difference $U_k = \Theta_k - \varphi_0$ that depends on "neutral point displacement" now falls at each air gap k, so that

$$\Phi_1 = (\Theta_1 - \varphi_0)/r_1 \tag{7.69}$$

It is clear that in energy equation (7.67), neither Φ nor U_m can be considered constant with changes in r_m.

The systems similar to that depicted in Fig. 7.12 enjoy wide use in practice for magnetic suspension. Stable equilibrium of a moving element is ensured by quick-acting current regulators. With electromagnets operated on alternating current, it is expedient to insert capacitors such that the electromechanical system containing capacitances and inductances can be at resonance; motion of a moving element (the wheel in Fig. 7.12) may cause a required change of currents in the coils as the system departs from or approaches the condition of resonance.

In the magnetic circuit such as that of Fig. 7.12 containing the coils (not shown) wound on the cores and connected by any suitable method to capacitors, all the unknown currents in the circuit branches are found to be related to the unknown magnetic fluxes of the magnetic circuit. The calculation of such circuits, therefore, often proves to be rather involved.

Chapter 8 Varying Electromagnetic Field in a Conducting Medium. Skin Effect

8.1. General

Time-varying electromagnetic fields are described by Maxwell's equations. The simultaneous solution of the two basic equations for dielectrics is quite different in character from that for conductors because these two types of material differ in the relative share of irreversible dissipation of electromagnetic energy, generally known as loss.

In the case of linear media and in the absence of moving bodies, it is possible to rely on the analysis of sinusoidally varying steady-state fields, without losing generality, and write the equations in complex notation:

$$\operatorname{curl} \dot{\mathbf{H}} = (\sigma + j\omega\varepsilon\varepsilon_0)\dot{\mathbf{E}} \tag{8.1a}$$

$$\operatorname{curl} \dot{\mathbf{E}} = -j\omega\mu\mu_0\dot{\mathbf{H}} \tag{8.1b}$$

Excluding from this system of equations one of the two unknowns ($\dot{\mathbf{E}}$ or $\dot{\mathbf{H}}$), we pass to one equation in one unknown ($\dot{\mathbf{H}}$ or $\dot{\mathbf{E}}$) by raising the order of derivatives.

Taking the curl of both sides of Eq. (8.1a) yields

$$\operatorname{curl}\operatorname{curl} \dot{\mathbf{H}} = (\sigma + j\omega\varepsilon\varepsilon_0)\operatorname{curl} \dot{\mathbf{E}} \tag{8.2}$$

if in the region of interest the medium parameters such as σ and ε are independent of coordinates, i.e. $\nabla\sigma = \nabla\varepsilon = 0$.

After substitution of $\operatorname{curl} \dot{\mathbf{E}}$ from (8.1b), the equation has the only unknown $\dot{\mathbf{H}}$ left:

$$\operatorname{curl}\operatorname{curl} \dot{\mathbf{H}} = -(\sigma + j\omega\varepsilon\varepsilon_0)j\omega\mu\mu_0\dot{\mathbf{H}} \tag{8.3}$$

A similar equation results after eliminating $\dot{\mathbf{H}}$:

$$\operatorname{curl}\operatorname{curl} \dot{\mathbf{E}} = -(\sigma + j\omega\varepsilon\varepsilon_0)j\omega\mu\mu_0\dot{\mathbf{E}} \tag{8.4}$$

The left-hand sides of the last two expressions contain the operation

$$\operatorname{curl}\operatorname{curl} \dot{\mathbf{F}} \equiv -\nabla^2\dot{\mathbf{F}} + \operatorname{grad}\operatorname{div} \dot{\mathbf{F}} \tag{8.5}$$

If the properties of a medium do not depend on the coordinates and, besides, the space charges are absent,

$$\operatorname{grad}\operatorname{div} \dot{\mathbf{F}} = 0 \tag{8.6}$$

for $\operatorname{div} \dot{\mathbf{E}}$ and $\operatorname{div} \dot{\mathbf{H}}$ are equal to zero under these conditions. Eqs. (8.3) and (8.4) then simplify yet more to

$$\nabla^2\dot{\mathbf{F}} = (\sigma + j\omega\varepsilon\varepsilon_0)j\omega\mu\mu_0\dot{\mathbf{F}} \tag{8.7}$$

where $\mathbf{F}$ is the field strength vector of either the magnetic field, $\dot{\mathbf{H}}$, or electric field, $\dot{\mathbf{E}}$.

On the right side of the expression, the complex factors $j\omega$ and $(j\omega)^2$ symbolically express the first and the second time derivatives, $\partial/\partial t$ and $\partial^2/\partial t^2$.

The form of Eq. (8.7) for describing a particular process depends on the parameters of a medium. Of most importance are two limiting cases: an ideal dielectric and a good conductor.

Ideal dielectric. Wave equation. In an ideal dielectric where $\sigma = 0$, Eq. (8.7) describes the propagation of a wave in a lossless medium:

$$\nabla^2\dot{\mathbf{F}} = (j\omega)^2(1/v^2)\dot{\mathbf{F}} \tag{8.8}$$

where $v = 1/\sqrt{\varepsilon\varepsilon_0\mu\mu_0}$ is wave velocity.

The differentiation with respect to time can also give the expression in explicit form:

$$\nabla^2\mathbf{F} = (1/v^2)(\partial^2\mathbf{F}/\partial t^2) \tag{8.9}$$

In the simplest case, when $\mathbf{F}$ depends on a single space coordinate, say, the z coordinate, and there is only any one of the components of $\mathbf{F}$, expressions (8.8) and (8.9)

simplify into the form

$$d^2F_k/dz^2 = (j\omega)^2(1/v^2)F_k \tag{8.10}$$

$$\partial^2F_k/\partial z^2 = (1/v^2)(\partial^2F_k/\partial t^2) \tag{8.11}$$

These are the equations of circuit theory that apply to transients (waves) in lossless lines.

A disturbance such as a signal of voltage u and period T that has occurred at the start of the line propagates along this line at a speed v. The signal will arrive at the point A, a distance a remote from the origin of disturbance (Fig. 8.1), only in time a/v; this is the time required for the wave to cover the path a. As soon as the wave passes by the point A, the signal at the point vanishes. A certain time after the wave has moved away, a return signal may appear as does echo. This pattern of wave propagation is typical for any physical process of wave nature rather than only for the electromagnetic process.

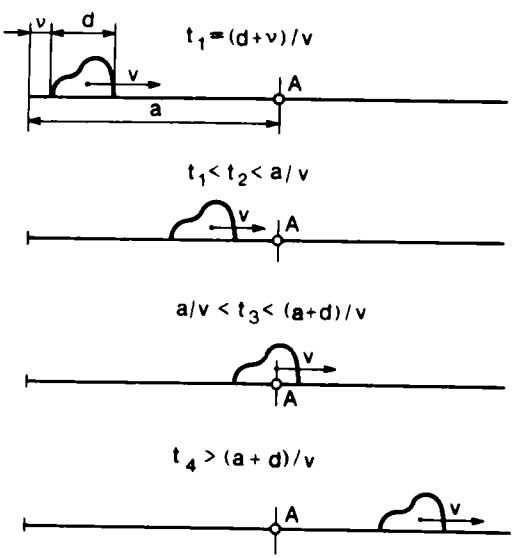


Fig. 8.1

The propagation of electromagnetic waves (and their reflection) in dielectrics is dealt with in Chs. 9 and 10.

Good conductors. Diffusion equation. In good conductors, σ is much higher than $\omega\varepsilon\varepsilon_0$. This is another limiting case for Eq. (8.7). The condition of good electric conduction is equivalent to the statement that in the medium under study the conduction currents dominate over displacement currents. Eq. (8.7) then takes the form

$$\nabla^2\dot{\mathbf{F}} = j\omega\mu\mu_0\sigma\dot{\mathbf{F}} \tag{8.12}$$

This equation can be rewritten to include explicit time derivatives:

$$\nabla^2\mathbf{F} = \mu\mu_0\sigma\partial\mathbf{F}/\partial t \tag{8.13}$$

Equations (8.12) and (8.13) describe the electromagnetic processes in a well conducting medium. These processes are treated in the subsequent sections of this chapter.

Where it is possible to consider only one of the components of vector $\dot{\mathbf{F}}$ in the cartesian coordinate system (for example, where $F = F_x$ at $F_y = F_z = 0$), Eq. (8.13) becomes simpler because now it describes a scalar

$$\nabla^2 F = b\partial F/\partial t \tag{8.14}$$

where $b = \mu\mu_0\sigma$.

The equation of this form is similar to the equations of diffusion or heat conduction. To understand better the procedure required for the solution of the equations similar to the above one, it is worthwhile to discuss in brief the solution to the equations describing diffusion proper.

Diffusion processes are dealt with in a number of branches of physics. In general, they can be described thus: if a certain medium has its homogeneity disturbed because of the accumulation of particles with higher volume density n, these particles move in the direction of particles with lower density tending to spread over uniformly in the space. The speed of particles, $\mathbf{w}$, is proportional to the density gradient negative in sign:

$$\mathbf{w} = -\gamma \operatorname{grad} n \tag{8.15}$$

The vector $\mathbf{w}$ is analogous to the vector of current density; it is equal to the number of particles transferred in a unit time through a unit area of the surface normal to the vector $\mathbf{w}$.

If the particles under examination do not vanish, nor originate, their density changes only as a consequence of their motion. The divergence of speed $\mathbf{w}$ is then equal to the rate of decrease of the particle density:

$$\operatorname{div} \mathbf{w} = -\partial n/\partial t \tag{8.16}$$

Notice once more the analogy to the relation between the density of current and that of space charge (the law of charge conservation).

After the substitution of $\mathbf{w}$ from (8.15) into (8.16), we obtain the differential diffusion equation

$$\gamma \operatorname{div} \operatorname{grad} n = \partial n/\partial t$$

or

$$\nabla^2 n = b\ \partial n/\partial t \tag{8.17}$$

where $b = 1/\gamma$. From the mathematical viewpoint, this equation differs from (8.14) only in letter designations.

We can get the same equation if we consider the processes of temperature distribution, namely, the process of heat transfer due to thermal conduction.

Comparison of heat equations with electromagnetic field equations allows us to extend the clear and illustrative concepts of temperature distribution into the field of electromagnetism. It is even sometimes possible to use just the obtained solutions, or anyway the methods suggested by Fourier in his book *Théorie analytique de la chaleur.*

The vector of thermal flux density $\mathbf{Q}$ is opposite in direction to the temperature gradient θ:

$$\mathbf{Q} = -k \operatorname{grad} \theta \tag{8.18}$$

where k is thermal conductivity.

The thermal energy density at any point is expressed as the product of temperature θ and thermal capacity c. If there is no heat generation in a medium (i.e. no increase in the thermal energy density due to chemical processes or heating of the substance by electric current), then it is only the heat of flux $\mathbf{Q}$ that changes the thermal energy density. Expressing the density of thermal (free) energy as the product of thermal capacity and temperature, we obtain the formula

$$c\partial\theta/\partial t = -\operatorname{div}\mathbf{Q} \tag{8.19}$$

which says that the rate of rise in thermal energy density, $c\partial\theta/\partial t$, is equal to the rate of heat inflow $-\operatorname{div}\mathbf{Q}$. When the divergence is positive, the temperature drops since the outflow of heat, $\operatorname{div}\mathbf{Q} > 0$, occurs as a result of the decrease in thermal energy density (see Sec. 7.2).

Substituting (8.18) into (8.19) gives

$$\nabla^2\theta = b\partial\theta/\partial t \tag{8.20}$$

where $b = c/k$. This expression differs from (8.14) and (8.17) only by the value of coefficient b.

Equations (8.15) and (8.16) can readily be solved for the speed vector $\mathbf{w}$ as well. For this, it is necessary to differentiate (8.15) with respect to time and substitute $\partial n/\partial t$ from (8.16) into the solution. Finally, we find that

$$\partial\mathbf{w}/\partial t = \gamma\operatorname{grad}\operatorname{div}\mathbf{w} = \gamma\nabla^2\mathbf{w}$$

Similarly, differentiating (8.18) with respect to time and substituting $\partial\theta/\partial t$ from (8.19), for thermal processes we get

$$\partial\mathbf{Q}/\partial t = (k/c)\nabla^2\mathbf{Q}$$

The two latter equations bear a still more resemblance to Eq. (8.13) for the electromagnetic field in a conducting medium. Unfortunately, there are few simple analogies as regards boundary conditions in order to achieve the similarity of solutions.

Analogy between thermal and electromagnetic processes. Owing to our tactile sense, we have a certain experience that enables us to judge of the process of heat diffusion. We know that a kettle placed on fire does not boil at once, a thick porcelain cup remains cool some time after pouring hot water into it, etc. We can get a much clearer idea of the rate at which the magnetic field sets in in a conducting medium if we consider geometrically similar thermal and electromagnetic systems under similar initial conditions. Of course, the numerical value of factor b in (8.14) and (8.20) is of much importance. However, it is possible to change the scale, i.e. units of measurement, and measure the time in seconds in one case and in milliseconds in the other, etc. The higher the value of b, the slower the process of transition to the steady state.

Table 8.1 gives the values of the constant b for some materials.

The following systems prove similar at comparable values of b: (1) a metallic or porcelain rod with an initial temperature $\theta(0)$ quickly immersed in a medium of temperature θ_0 (Fig. 8.2) and (2) a steel rod (but neither a copper nor a porcelain bar) placed inside a solenoid (Fig. 8.3), in which at moment $t = 0$ a current i_0 begins to flow and a field H_0 appears in the space around the rod; before then, the field inside

Table 8.1

b, s/cm^2	Steel $\mu = 2\,000$	Steel $\mu = 10\,000$	Copper	Lead	Porcelain
c/k	5.5	5.5	0.90	4.0	12.5
$\mu\mu_0\sigma$	2.5	12.5	0.007 2	0.000 6	10^{-15}

the rod was equal to zero, $H(0) = 0$. This electromagnetic system has only one magnetic field component parallel to the solenoid axis. That is why the process is described by the scalar equation (8.14).

Instead of solid rods, we can consider tubes of external radius a and wall thickness l. A tube shields the interior both from the heat of the medium with a temperature θ_0 and from the magnetic field H_0.

In similar systems, the same equations (differing only in designations, θ or H) are applicable to identically bounded regions. Such systems show the similarity of boundary conditions $\theta = \theta_0$ or $H = H_0$ at $r = a$ and initial conditions $\theta = \theta(0)$ or $H = H(0)$ at $r < a$.

With the same geometrical dimensions and the same values of b, it is obvious that

$$h(r, t) = \nu(r, t) \text{ at } r \leqslant a \text{ and } t > 0 \tag{8.21}$$

where

$$h = H/H_0 \text{ and } \nu = [\theta - \theta(0)]/[\theta_0 - \theta(0)]$$

Figure 8.4 illustrates the plots of h versus r/a for different moments of time at $b = 12.5$ s/cm^2 and a rod radius $a = 1$ cm (see Figs. 8.2 and 8.3). The curves show that a steel rod of a radius of 1 cm and $\mu = 10\,000$ gradually gets magnetized when exposed to a field $H_0 = 1$ A/m; the field strength on the rod axis becomes equal to about 0.1 H_0 in 1 s and then reaches 0.7 H_0 only in 3.37 s.

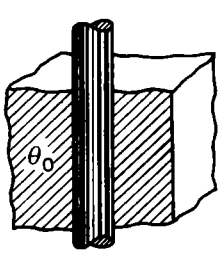


Fig. 8.2

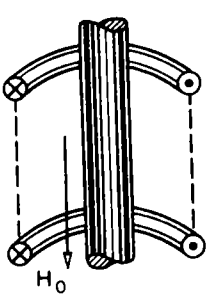


Fig. 8.3

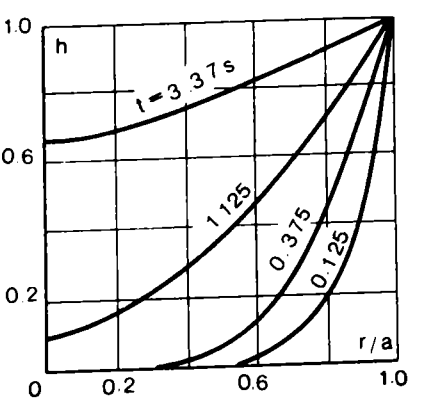


Fig. 8.4

A porcelain rod of identical dimensions heats up at the same rate, with the same temperature distribution throughout its bulk: when immersed in boiling water the rod with an initial temperature of 20°C shows an increase of temperature along its axis by merely 8°C in 1 s. If the values of b and a are other than the ones given above, the same curves of Fig. 8.4 can be suitable if we replace the time specified for each curve by the time

$$t' = t(b'/b)(a'/a)^2 \tag{8.22}$$

where b' and a' are new values of the parameter b and radius a. For example, the field will permeate a copper cylinder of radius $a = 0.5$ cm at a greater rate: the field on its axis will reach 70% of the external field in a time $t' = 0.48$ ms. Its heating will take a longer time because $c/k > \mu\mu_0\sigma$ for this cylinder.

The shielding effect of the external layers of a solid cylinder (or the walls of a tube) during heating is due to the opposition the layers offer to the thermal flux which must also heat these layers. That is why a material of higher thermal capacity heats slower. The higher the thermal conductivity k of a material, the greater the rate of its heating.

For the electromagnetic case, the causes of the shielding effect of a conducting tube are certainly different. An external magnetic field applied to a tube (or a solid cylinder) induces an emf in its layers as the field penetrates deeper into the material bulk. By Lenz' law the induced emf produces a current that opposes a change in the magnetic flux linking the loop; in the given case, the induced current produces a counter field that prevents the applied field from penetrating into the tube or rod bulk.

Although the physical causes that slow down the rate of field and temperature distribution are different, the analogy can fairly be close. In solving engineering problems, therefore, analogous approaches are sometimes used. For example, to ensure a fast penetration of the magnetic field into steel cores, they are usually assembled from thin insulated laminations (transformer cores). The same approach finds use in heat engineering for constructing radiators.

An electric field also meets with opposition as it strives to penetrate into the bulk of a metal. This effect is particularly evident in conductors carrying alternating current (see Sec. 8.4).

Varying (ac) field in a conducting medium. The most frequent case dealt with in practice relates to the processes of field penetration into a conducting medium when exposed to a simple sinusoidally varying field. Here we meet with certain peculiarities in the solution of Eq. (8.12), which is very important to undestand from the physico-mathematical viewpoint.

Consider the case of heating the rod of Fig. 8.2 assuming that the environment temperature varies harmonically:

$$\theta_0 = \theta_m \sin \omega t + \theta_1 \tag{8.23}$$

It fluctuates about a mean value, θ_1. The rod temperature θ gradually rises with θ_0, the outer layers being the first to acquire a higher temperature.

If θ_0 were to change slowly, the entire rod would have time to heat up to the environment temperature, i.e. to θ_0. But since θ_0 changes fast, only a thin outer layer is able to heat through to a temperature close to θ_0. We can readily picture the

temperature changes as being so fast that the internal layers heat above θ_1 only by the moment when the environment temperature begins to drop, falling below the mean value ($\theta_0 < \theta_1$). The incomplete process of heating thus changes for the process of cooling that, in turn, only gradually affects the bulk of the material. But with rather rapid variations in ambient temperature, the inner layers have no time to cool or heat rather heavily; substantial temperature changes (with an amplitude close to θ_m) occur only in outer layers, and the amplitude of temperature variations gradually decreases with distance into the material bulk.

We should take notice of the fact that the rate of heating to the highest level (the amplitude of temperature variation) is faster in external than in internal layers. It is as if the temperature propagated in a wavelike manner: the crest of the wave moves from the surface to the axis, gradually decreasing in height (wave decaying); the process of propagation may also be thought of as phase lag with the propagation of the wave into the body bulk.

Skin effect during heat diffusion. In the example considered above the skin effect shows up quite vividly: a varying disturbance in the surrounding medium does not penetrate into the bulk of massive bodies; the higher the frequency of a disturbance the lower the depth to which it affects a material.

An illustrative example of the effect of frequency on the depth of penetration is a change in the temperature of soil whose surface is subject to periodic yearly and daily temperature fluctuations. Daily temperature changes affect the soil to a much smaller depth than yearly changes. For this case, we can also visualize a phase lag of temperature variations in soil layers with an increase in the depth of penetration. As is obvious, at the onset of frost it is only the upper layers of soil that freeze through, and in early spring only the surface layers of soil start to thaw out. It is interesting that at a certain depth, the highest temperature is found to be only in winter (only by this period can summer heat penetrate to this depth) and the lowest temperature in summer.

Let us note once more that although heat diffusion and electromagnetic field penetration are formally similar, the physical processes that cause a delay of external influences are substantially different.

Quasiwave propagation in a resistance-capacitance (rC) cable. For the calculation of a cable having a relatively large resistance and appreciable capacitance, Kelvin suggested an approximate solution of equations by assuming that the inductance L_0 and conductance g in the transverse cross section are zero. The basic equations that describe the process occurring along such a resistance-capacitance cable (along the z axis) are

$$\begin{aligned} d\dot{U}/dz &= -r_0\dot{I} \\ d\dot{I}/dz &= -j\omega C_0\dot{U} \end{aligned} \tag{8.24}$$

The simultaneous solution of these equations for, say, voltage takes the form

$$d^2\dot{U}/dz^2 = j\omega r_0 C_0 \dot{U} \tag{8.25}$$

This expression differs from diffusion equation (8.12) only in that it describes the process in the cable in the single space coordinate. Since the time dependence is given

in complex notation, the equation is written in common rather than in partial derivatives. Its solution takes the form

$$\dot{U} = A_1 e^{-\gamma z} + A_2 e^{\gamma z} \tag{8.26}$$

where $\gamma = \alpha + j\beta = \sqrt{j\omega r_0 C_0}$; $\alpha = \beta = \sqrt{\omega r_0 C_0/2}$; and A_1, A_2 are integration constants. Indeed, Eq. (8.26) satisfies Eq. (8.25), which is easy to confirm by performing the substitution.

Expression (8.26) represents a sum of two waves traveling in oposite direction at phase velocities

$$v_{ph} = \pm \omega/\beta = \pm c/\sqrt{\varepsilon\mu} \tag{8.27}$$

In the case of a sinusoidally varying supply source, a decaying wave progresses along the resistance-capacitance cable with a definite phase velocity. This is the process that follows the diffusion equation or, more definitely, the process that obeys the equation for an electromagnetic field in a conducting medium (and even in a well conducting medium). Why do then we call this process the quasiwave, or seemingly wave process?

Notes on wave processes. In the strict sense of the term, a wave process is the process which follows the equations

$$f(t, R) = f(t - R/v) \tag{8.28}$$

$$f_1(t, R) = m(R) f(t - R/v) \tag{8.29}$$

in which R is the coordinate in the direction of wave propagation; v is the wave velocity; and $m(R)$ is a function that defines the wave decaying and depends on the path covered by the wave (i.e. on R).

If $f(t - R/v)$ is different from zero only in a certain interval of the values of an argument

$$v_1 \leqslant t - R/v \leqslant v_2 \tag{8.30}$$

then at point R_1 this function is nonzero in the time interval $\tau_1 \leqslant t \leqslant \tau_2$ in accordance with (8.30); here τ_1 is the instant at which the wave front has arrived, and τ_2 is the instant at which the wave tail has gone.

Quasiwave equations and corresponding solutions do not give at all the picture of a wave process described by (8.28) and (8.29). The main cause of the discrepancy lies in that quasiwave processes follow the diffusion equation only in the case of a periodic disturbance and only where the solution is to be found for a steady state; in these processes, the phase constant β and attenuation constant α, and also the phase velocity, depend on frequency.

As regards an aperiodic process for example a signal (see Fig. 8.1), its spectral decomposition shows a set of the continuous series of frequencies. Since in a quasiwave process both the attenuation and velocity and also the phase depend on frequency, the signal undergoes distortion in shape during its propagation.

It is of interest to note that phase velocity grows with frequency, but then attenuation increases as well. That is why a harmonic component of infinitely high frequency ($\omega \to \infty$) propagates at an infinitely high velocity, but attenuates at an infinite rate ($e^{-\alpha z} \to 0$ when $\alpha \to \infty$). This is the reason why a signal transmitted from, say, one point of the line instantly arrives at another point at the end of the line, but with a

zero amplitude; a disturbance at the end of the line grows but gradually. It is exactly the pattern of heat distribution from the periphery to the axis of the rod or propagation of an instantly produced external magnetic field (see Figs. 8.2 and 8.3).

Waves in a medium with equivalent complex parameters. In Ch. 3 we have shown by way of Eqs. (3.54) and (3.57) how we can pass to equivalent complex parameters $\underline{\sigma}_{eq} = \underline{\sigma}$ or $\underline{\varepsilon}_{eq} = \underline{\varepsilon}$ between which a simple relation exists:

$$\underline{\sigma} = j\omega\varepsilon_0\underline{\varepsilon} \tag{8.31}$$

Does this mean that the introduction of complex parameters eliminates the difference between the wave and the diffusion equation? As a matter of fact, the factor on the right of the field equation can be represented in any of the two equivalent forms:

$$\nabla^2\dot{\mathbf{F}} = (j\omega)^2\varepsilon_0\underline{\varepsilon}\,\mu_0\underline{\mu}\dot{\mathbf{F}} \tag{8.32}$$

or

$$\nabla^2\dot{\mathbf{F}} = j\omega\underline{\mu}_0\mu\underline{\sigma}\,\dot{\mathbf{F}} \tag{8.33}$$

We shall restrict ourselves to the analysis of a simple case where $\dot{\mathbf{F}}$ depends on the single cartesian coordinate z and, hence

$$\nabla^2\dot{\mathbf{F}} \equiv d^2\dot{\mathbf{F}}/dz^2 \tag{8.34}$$

The solution of both (8.32) and (8.33) can then be written in the form

$$\dot{\mathbf{F}} = \mathbf{A}_+ e^{-\alpha z}e^{-j\beta z} + \mathbf{A}_- e^{\alpha z}e^{j\beta z} \tag{8.35}$$

where the propagation constant

$$\gamma = \alpha + j\beta = j\omega\sqrt{\underline{\varepsilon}\varepsilon_0\underline{\mu}\mu_0} = \sqrt{j\omega\mu\mu_0\underline{\sigma}} \tag{8.36}$$

If the condition (8.31) is valid, both expressions for γ are obviously equivalent. What makes the wave and quasiwave processes differ is not certainly the notation used for the equations, but the properties of a medium: Eqs. (8.32) and (8.33) describe a purely wave process in a loss-free medium where $\underline{\mu} = \mu$ and $\underline{\varepsilon} = \varepsilon$. The same equations express a quasiwave process (obeying the diffusion equation) in a medium where $\underline{\sigma} = \sigma$ and $\underline{\varepsilon} = \sigma/j\omega\varepsilon_0$, i.e. in a conductor.

In the first case, Eqs. (8.32) and (8.33) are valid for the wave process specified by (8.28) at $R = z$; a wave progresses at a constant speed $v = c/\sqrt{\mu\varepsilon}$, where $c = 1/\sqrt{\mu_0\varepsilon_0}$ is the light velocity in free space. In the second case, the notion of wave propagation is adequate only if a disturbance is periodic in character. Where an aperiodic disturbance takes place, we cannot speak of any waves at all.

In the general case, all the parameters $\underline{\varepsilon}$, $\underline{\mu}$ and $\underline{\sigma}$ are complex, and the general solution gives a decaying wave with a phase velocity and attenuation that are dependent on frequency. But if the wave velocity and attenuation are frequency dependent (wave dispersion takes place), it is apparent that a propagating disturbance (signal) is subject to distortion; expressions (8.28) and (8.29) are then no longer valid. However, losses (μ'', ε'', and σ) in a medium are often very small, and so the dispersion (the dependence of wave velocity and attenuation on frequency) is accordingly small too. The propagation of electromagnetic waves then practically obeys expressions (8.28) and (8.29).

One more point of interest is that as the frequency rises, $\underline{\mu}$ and $\underline{\varepsilon}$ always approach a

finite value of not less than unity, whatever the kind of frequency dependence of these quantities. In the limit, therefore, the wave velocity at $\omega \to \infty$ reaches the velocity of light in a vacuum.

In Ch. 9, we shall consider the simple cases of the solutions to wave equations for given boundary conditions (propagation and reflection of waves); it will be shown that where the field depends only on one coordinate, the known methods of calculation of long waves are particularly suitable.

Electromagnetic compression. In conclusion, let us consider an interesting example of the practical use of a varying magnetic field for heavy compression of metal parts, fitting, for example, of a copper sleeve on a steel shaft. The method relies on the creation of a short impulse of a magnetic field that has no time to penetrate into the bulk of metal; in other words, the method essentially uses the skin effect.

Assume the field is set up by a pulsating current in the turns of a solenoid that encircles a steel rod with a copper belt to be tightly pressed on to it (Fig. 8.5). As mentioned above, the rapidly growing field induced by a short pulse of current has no chance to penetrate into the bulk of copper. One of the approaches to the creation of pulsating current is to discharge a capacitor of a rather high capacitance and high voltage into the solenoid circuit. Such a simple arrangement is quite capable of producing fields of 300 kA/cm and higher.

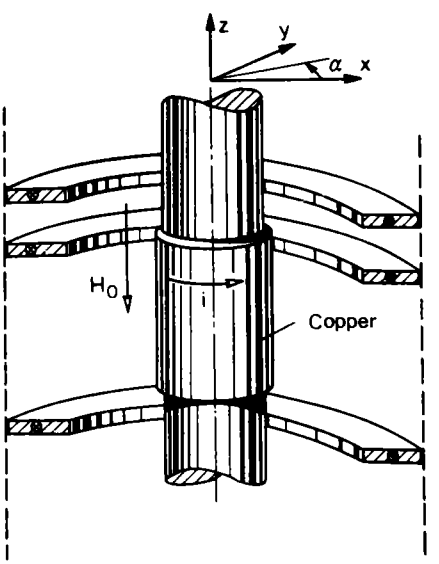


Fig. 8.5

For simplicity of calculations, let us assume that the H field in the surface layer of the belt of thickness $\rho = a - r_0 \ll r_0$ decays linearly:

$$H = H_z = -H_0(r - r_0)/(a - r_0)$$

The density of current induced in the belt surface layer is found from Maxwell's first equation (in the cylindrical coordinate system, see Appendix A5)

$$\mathbf{J} = \operatorname{curl}\mathbf{H} = -\mathbf{e}_\alpha \partial H_z/\partial r = \mathbf{e}_\alpha H_0/(a - r_0)$$

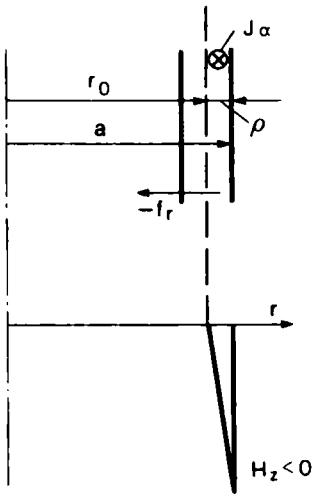


Fig. 8.6

This current density is easy to evaluate by differentiating in the cylindrical coordinates at $\partial/\partial\alpha = \partial/\partial z \equiv 0$. Considering the direction of currents, $J = J_\alpha$, we see that it is the current density component J that opposes the field penetration.

It remains to estimate the force per unit area of the cylinder surface. The force per unit volume due to the interaction between the current and field is $\mathbf{f} = \mathbf{J} \times \mathbf{B}$. By integrating this force directed toward the rod axis (Fig. 8.6) along the radius, we find the pressure, i.e. the force per unit area of the surface that is normal to this surface:

$$F = \int_{r0}^{a} (-f_r)dr = \int_{r0}^{a} \mu_0 H_0^2 (r - r_0)/(a - r_0)^2 dr = \mu_0 H_0^2/2 \qquad (8.37a)$$

This formula is valid, whatever the manner in which H changes in the copper belt, if the field on the inner surface of the belt reduces to zero. We can readily prove the point if we do not hasten to start with differentiating and work out the solution up to the end in the general form. Indeed, at $H = H_z(r)$, i.e. when $\mathbf{H}$ is parallel in direction to the axis and is only dependent on the radius,

$$\mathbf{J} = \operatorname{curl}\mathbf{H} = -\mathbf{e}_\alpha \partial H_z/\partial r$$

The force is then

$$f_r = \mu_0 J_\alpha H_z = -\mu_0 H_z \partial H_z/\partial r$$

and hence

$$F = \int_{ri}^{a} (-f_r)dr = \mu_0 \int_{ri}^{a} H_z(\partial H_z/\partial r)dr = \mu_0 \int_{Hi}^{H0} H_z dH_z = \mu_0[(H_0^2 - H_i^2)/2] \qquad (8.37b)$$

In these formulas, the subscript i stands for the internal cylinder surface. If $H_i = 0$, then $F = \mu_0 H_0^2/2$ as pointed out earlier.

The quantity F that expresses the field energy density can be regarded as a field pressure on a surface (tangent to the field vector).

The effect similar to the one described above is also evident when a heavy current flows in a conductor. For example, the copper shell of a tubular lightning arrester was once flattened under a lightning stroke, more exactly, pressed to the inner steelwork. The current through the structure was found from the pressure sufficient to flatten the shell. The calculation is quite analogous to that described above. Strictly speaking, the result is adequate only on the assumption that the stress on a solid body exceeds the limit of electricity, otherwise the calculation should be made exactly by the rules of the theory of elasticity.

8.2. Varying Magnetic Field in a Thin Plate

Transformer core steel laminations. The field in a core plate is the simplest and, at the same time, the widespread case. The core plates are insulated one from another and each has no effect on the other, at least in a first approximation. The width of a plate, b, is generally by far larger than its thickness, $2a$.

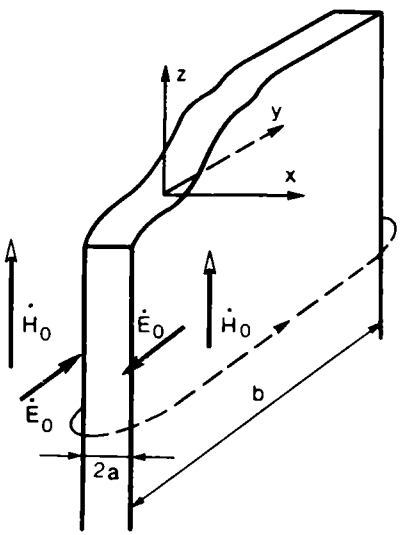


Fig. 8.7

Let us dispose the axes of cartesian coordinates so that the magnetic field H_0 exterior to the plate should have only the zth component (Fig. 8.7); this field is the same in strength on both sides of the plate. Assuming that the plate is normal to the x axis, we can assert that the H field has only the zth component and depends only on the x coordinate inside the homogeneous and isotropic plate. The basic field equation (8.12)

$$\nabla^2\dot{\mathbf{H}} = j\omega\mu\mu_0\sigma\dot{\mathbf{H}}$$

then simplifies considerably and takes the form

$$d^2\dot{H}/dx^2 = \gamma^2\dot{H} \qquad (8.38a)$$

at

$$\gamma = \alpha + j\beta = \sqrt{j\omega\mu\mu_0\sigma} \tag{8.38b}$$

where $\dot{H}$ is certainly equal to $\dot{H}_z$.

Propagation of plane waves. Differential equation (8.38a) does not differ from (8.25) for a resistance-capacitance cable; its solution is known:

$$\dot{H} = A_1 e^{-\gamma x} + A_2 e^{\gamma x} \tag{8.39}$$

The resultant field consists of two waves traveling in opposite directions $\pm x$ at a phase velocity

$$v_{ph} = \omega/\beta \tag{8.40}$$

and a wavelength

$$\lambda = v_{ph}T = v_{ph}/f = 2\pi/\beta \tag{8.41}$$

When the field strength on both sides of the plate is the same and equal to $\dot{H}_0$, the constants are determined from identical boundary conditions if the origin of coordinates is located in the middle of the plate:

$$\begin{aligned} \dot{H}_{x=a} &= \dot{H}_0 = A_1 e^{-\gamma a} + A_2 e^{\gamma a} \\ &= \dot{H}_{x=-a} = A_1 e^{\gamma a} + A_2 e^{-\gamma a} \end{aligned} \tag{8.42}$$

From these formulas it follows that

$$A_1 = A_2 = \dot{A} = \dot{H}_0/(e^{\gamma a} + e^{-\gamma a}) = \dot{H}_0/2\cosh\gamma a$$

After substituting the constants so found in (8.39), we obtain

$$\dot{H} = \dot{H}_0(\cosh\gamma x/\cosh\gamma a) \tag{8.43}$$

The magnetic field distribution over the plate thickness is shown in Fig. 8.8, where the normalized field strength, $h = H/H_0$, is plotted versus the normalized coordinate x/a[1]:

$$h(x) = |\dot{H}/\dot{H}_0| = |\cosh\gamma x/\cosh\gamma a| \tag{8.44}$$

In Fig. 8.8, the function $h(x)$ is plotted for four values of αa. The values 0.5 and 1.0 correspond to the normal operating conditions of transformer cores employed in power systems. As seen from (8.38b), a twofold increase in αa corresponds to a twofold increase in the plate thickness or fourfold increase in frequency (or permeability, or conductivity, or, finally, product $\omega\mu\sigma$).

The thickness of transformer steel sheets for work at a commercial frequency is 0.35 mm; the curve $\alpha a = 1$ of Fig. 8.8 represents the field strength distribution in such a sheet at 50 Hz, and the curve $\alpha a = 4$ at 800 Hz. The transformers intended for higher frequencies and also those designed to operate under pulse conditions have the cores made from thin permalloy tapes about 10^{-4} cm in thickness. The high-speed elements of automatic equipment use thin films 4×10^{-6} cm to 4×10^{-5} cm

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[1] If the tables of the hyperbolic functions of the complex argument are not available, use should be made of the known formula

$$\cosh(x + jy) = \cosh x \cdot \cos y + j \sinh x \cdot \sin y$$

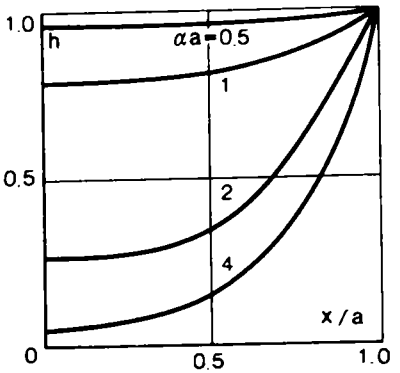

Fig. 8.8

in thickness. The surface effect in such films is negligible.

Representing the field (8.43) as a sum of two waves moving in opposite directions,

$$\dot{H} = (\dot{H}_0/2\cosh\gamma a)e^{-\gamma x} + (\dot{H}_0/2\cosh\gamma a)e^{\gamma x} = \dot{H}_+ + \dot{H}_- \qquad (8.43a)$$

we can assert that it is safe to neglect one wave if the real part of the exponent is large in value. Thus, when $\alpha a = 4$ (see Fig. 8.8) for points $x = 0.5a$, the ratio between the amplitudes of waves moving in opposite directions is $|\dot{H}_-/\dot{H}_+| = e^4 = 54.6$.

In (8.43a), $\dot{H}_-$ represents the wave that travels in the negative x direction (from right to left in Figs. 8.7 and 8.9). This wave has covered the path from the plate surface to the point $x = 0.5a$, which is equal to $0.5a$. In the same expression, $\dot{H}_+$ represents the wave traveling in the positive x direction (from left to right in Figs. 8.7 and 8.9); this wave has traveled the path from the left surface of the plate to the point $x = 0.5a$, the distance being equal to $1.5a$.

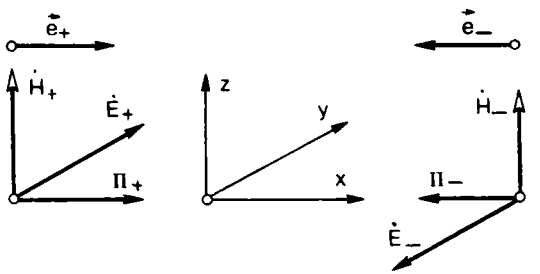

Fig. 8.9

Under these conditions the plate can be considered nontransparent to the electromagnetic wave of a given frequency. It can thus be assumed that only one wave enters the region $|x| \geqslant 0.5a$ in the direction from left to right of the plate. The ratio $h = |\dot{H}/\dot{H}_0|$ is then expressed just by the exponent

$$e^{-\alpha(a-x)} = e^{-\alpha a(1-x/a)}$$

The electric field $\dot{\mathbf{E}}$ (or the corresponding current density $\dot{\mathbf{J}} = \sigma\dot{\mathbf{E}}$) is easily found by using Maxwell's first equation

$$\dot{\mathbf{E}} = \operatorname{curl} \dot{\mathbf{H}}/\sigma$$

Differentiating (8.39) for plane waves, at $\dot{H} = \dot{H}_z$ we get

$$\dot{E} = \dot{E}_y = -(1/\sigma)(\partial \dot{H}/\partial x) \tag{8.45}$$

For $\dot{H}$ represented by (8.39) as a sum of two waves, the electric field takes the form

$$\dot{E} = \dot{E}_y = \zeta A_1 e^{-\gamma x} - \zeta A_2 e^{\gamma x} = \dot{E}_+ - \dot{E}_- \tag{8.46}$$

where

$$\zeta = \gamma/\sigma = \sqrt{j\omega\mu\mu_0/\sigma} = (1+j)\sqrt{\omega\mu\mu_0/2\sigma} \tag{8.47}$$

The field expression given above shows that for each wave taken separately, but not for the sum of two waves, the magnetic and the electric fields are interrelated: the electric field is equal to the magnetic field times ζ:

$$\dot{E}_+ = \zeta \dot{H}_+ = \zeta A_1 e^{-\gamma x} \tag{8.48a}$$

$$\dot{E}_- = \zeta \dot{H}_- = \zeta A_2 e^{\gamma x} \tag{8.48b}$$

The factor ζ is *the wave, intrinsic*, or *characteristic impedance of a medium*, which is measured in ohms. The direction of vectors $\dot{\mathbf{E}}_+$ and $\dot{\mathbf{E}}_-$ is related to the direction of vectors $\dot{\mathbf{H}}_+$ and $\dot{\mathbf{H}}_-$ and to the direction of wave propagation so that the corresponding Poynting vector $\boldsymbol{\Pi} = \mathbf{E} \times \mathbf{H}$ is in the direction of wave propagation (see Fig. 8.9):

$$\underline{\boldsymbol{\Pi}}_+ = \dot{\mathbf{E}}_+ \times \overset{*}{\dot{\mathbf{H}}}_+, \underline{\boldsymbol{\Pi}}_- = \dot{\mathbf{E}}_- \times \overset{*}{\dot{\mathbf{H}}}_- \tag{8.49}$$

The field vectors and unit vectors $\mathbf{e}_+$ and $\mathbf{e}_-$ that point to the direction of wave propagation are related to each other by the equality (see Fig. 8.9)

$$\dot{\mathbf{H}}_{+,-} = \mathbf{e}_{+,-} \times \dot{\mathbf{E}}_{+,-}/\zeta \tag{8.50a}$$

or

$$\dot{\mathbf{E}}_{+,-} = \zeta \dot{\mathbf{H}}_{+,-} \times \mathbf{e}_{+,-} \tag{8.50b}$$

For fields $\dot{\mathbf{H}}_{+,-}$ and $\dot{\mathbf{E}}_{+,-}$, the principle of superposition is applicable[1]:

$$\dot{\mathbf{H}} = \dot{\mathbf{H}}_+ + \dot{\mathbf{H}}_- \text{ and } \dot{\mathbf{E}} = \dot{\mathbf{E}}_+ + \dot{\mathbf{E}}_- \tag{8.51}$$

When taking the sum of components, however, it is certainly necessary to keep in mind the fact that vectors $\dot{\mathbf{E}}_+$ and $\dot{\mathbf{E}}_-$ are opposite in direction when $\dot{\mathbf{H}}_+$ and $\dot{\mathbf{H}}_-$ are in the same direction as shown in Fig. 8.9. The electric field is then

$$\dot{E} = \dot{E}_+ - \dot{E}_- = \zeta(\dot{H}_+ - \dot{H}_-) \tag{8.51a}$$

if

$$\dot{H} = \dot{H}_+ + \dot{H}_-$$

If waves propagate in opposite directions, the electric vectors $\dot{\mathbf{E}}_+$ and $\dot{\mathbf{E}}_-$ of two waves may happen to be in the same direction (see Sec. 8.5). The vectors $\dot{\mathbf{E}}$ and $\dot{\mathbf{H}}$ of one of the waves will then be at an angle of 180 ° with respect to each other, and so

[1] The superposition principle is not applicable to Poynting vectors, and the $\underline{\boldsymbol{\Pi}}$ vector of the resultant field is not equal to the sum of components $\underline{\boldsymbol{\Pi}}_+$ and $\underline{\boldsymbol{\Pi}}_-$ since the power is defined by quadrature components rather than by linear components.

the directions in which the waves progress will not change. Considering the components for this case, we have

$$\dot{E} = \dot{E}_+ + \dot{E}_- \text{ and } \dot{H} = \dot{H}_+ - \dot{H}_- \tag{8.51b}$$

Equalities (8.50) and (8.51) hold here. The expressions similar to (8.48) through (8.50) do not obviously apply to a resultant field.

Substituting $A_1 = A_2$ into (8.46) gives

$$\dot{E} = \zeta(\dot{H}_0/2\cosh\gamma a)(e^{-\gamma x} - e^{\gamma x}) = \zeta\dot{H}_0(\sinh\gamma x/\cosh\gamma a) \tag{8.52}$$

In a particular case where $x = \pm a$,

$$\dot{E}_0 = \mp\dot{E}_y = \zeta\dot{H}_0\tanh\gamma a \tag{8.52a}$$

We can also obtain the latter expressions by differentiating (8.43) in the same manner as we did in deriving (8.45). However, a slightly more intricate approach used here has given us a clearer idea of the decomposition of a resultant field into waves propagating in opposite directions. Also, we have become familiar with such an important parameter as the characteristic impedance of a medium and deduced useful expressions (8.48) through (8.51).

In the absence of a reflected wave and for the resultant field,

$$\dot{E} = \zeta\dot{H}$$

This relation is known as Leonovich's approximate boundary condition.

Average permeability. Knowing the external field strength $\dot{H}_0$ and the average magnetic induction across the plate[1]

$$\dot{B}_{av} = (1/2a)\int_{-a}^{a} \mu_0\mu\dot{H}dx = (1/a)\int_0^a \mu_0\mu\dot{H}dx = \mu_0\mu\dot{H}_{av} \tag{8.53}$$

we can determine the average permeability

$$\mu = \mu_{av} = \dot{B}_{av}/\mu_0\dot{H}_0 = \mu\dot{H}_{av}/\dot{H}_0 \tag{8.54}$$

As is clear from Fig. 8.8 and the entire analysis made above, μ_{av} can differ heavily from μ at large values of $|\gamma a|$.

It is not difficult to perform the above mathematical operations on condition that the plate is homogeneous. But the method described below has more points in its favor. It offers the possibility of evaluating the magnetic flux by Faraday's law:

$$-j\omega\dot{\Phi} = \dot{\mathscr{E}} = \oint\dot{\mathbf{E}}d\mathbf{l} \tag{8.55}$$

For the plate of Fig. 8.7, the integration along the closed path encircling the plate (the path being traced in the direction of vector **H**, or the z axis, i.e. in accordance with the corkscrew rule) leads to the equality

$$\oint\dot{\mathbf{E}}d\mathbf{l} = -2\dot{E}_0 b = -2b\zeta\dot{H}_0\tanh\gamma a \tag{8.56}$$

with allowance made for (8.52a)..

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[1] Of course, in performing the integration, we should consider the variation of the H phase as well as H with x.

Comparison of (8.56) with (8.55) gives

$$\dot{B}_{av} = \dot{\Phi}/2ab = (\zeta \dot{H}_0/j\omega a) \tanh \gamma a \tag{8.57}$$

and

$$\mu_{av} = \dot{B}_{av}/\mu_0 \dot{H}_0 = (\zeta/j\omega\mu_0 a) \tanh \gamma a \tag{8.58}$$

In Eq. (8.58) we can perform one more operation by substituting the expression for ζ given by (8.47):

$$\zeta/j\omega\mu_0 a = (\sqrt{\mu})/a\sqrt{j\omega\mu_0\sigma} = \mu/\gamma a$$

The expression for the average permeability of a plate now assumes the form

$$\mu_{av} = \mu_1 - j\mu_2 = \mu \tanh \gamma a/\gamma a \tag{8.59}$$

At low values of $|\gamma a|$, we can expand $\tanh \gamma a$ into a power series. Hence,

$$\begin{aligned} \mu_{av}/\mu &= 1 - (\gamma a)^2/3 + (2/15)(\gamma a)^4 - \ldots \\ &= 1 - (2/15)(\omega\mu\mu_0\sigma)^2 a^4 + \ldots - j\omega\mu\mu_0\sigma a^2/3 + \ldots \end{aligned} \tag{8.59a}$$

The average permeability is a complex quantity even if the permeability of the substance proper is a real number (magnetic loss is absent). The real part of the average permeability diminishes with a rise in $|\gamma a|$. At large values of $\mathrm{Re}(\gamma a)$, we may assume that $\tanh \gamma a \to 1$, and so

$$\mu_{av}/\mu = 1/\gamma a = 1/a\sqrt{j\omega\mu\mu_0\sigma} \tag{8.59b}$$

Expression (8.59b) corresponds to a "strong skin effect" or "high frequency" at which the plate remains nontransparent to an electromagnetic wave. On either side of the plate, a wave tends to propagate into the plate bulk:

$$\dot{H} = \dot{H}_0 e^{-\gamma n} \tag{8.60}$$

where n is the distance from the plate surface to a point in the plate bulk. The electric field strength on the plate surface is given by

$$\dot{E}_0 = \zeta \dot{H}_0 \tag{8.61a}$$

At such a high frequency, the inward flow of power through a unit area is

$$P_0 + jQ_0 = \dot{E}_0 \dot{H}_0^* = E_0^2/\zeta^* = H_0^2 \zeta \tag{8.61b}$$

or

$$P_0 = \mathrm{Re}(H_0^2 \zeta) = H_0^2\sqrt{\omega\mu\mu_0/2\sigma} = \mathrm{Re}(E_0^2/\zeta^*) \tag{8.61c}$$

In this case ($\tanh \gamma a \to 1$) the magnetic flux is directly proportional to the perimeter, as follows from (8.56). A further increase in the plate thickness does not result in a heavier magnetic flux because the magnetic field fails to penetrate deeper into the medium all the same and thus cannot cause complete magnetization of the plate.

Eddy current loss. The energy of a varying magnetic field irreversibly converted to a thermal energy per unit time is known as the dissipated power. In a ferromagnetic conductor where the imaginary part of permeability is zero, $\mu'' = 0$, all the dissipated power results from eddy currents, $J = \sigma E \neq 0$. By Joule's law, the amount of thermal power evolved in each element of volume is

$$dP = (1/\sigma)J^2 dV = E^2 \sigma dV \tag{8.62}$$

Integrating this expression throughout the entire thickness of the plate, we can find the loss due to eddy currents in the lamination, commonly expressed as power per unit volume or mass.

A more general definition of this type of loss reads: the complex power dissipated in a plate is equal to the power that flows into the plate through its side walls. This power is equal to the integral of the Poynting vector.

The flow of power per unit area of the plate of Fig. 8.7 is given by

$$\underline{\Pi}_0 = \dot{E}_0 \overset{*}{\dot{H}}_0 = \zeta \dot{H}_0 \tanh \gamma a \cdot \overset{*}{\dot{H}}_0 \tag{8.63}$$

and the flow of power per unit volume is

$$\underline{S}_0 = P_0 + jQ_0 = 2\underline{\Pi}/2a = H_0^2 \zeta (\tanh \gamma a)/a = H_0^2 j\omega\mu\mu_0 (\tanh \gamma a)/\gamma a \tag{8.64}$$

This power can be expressed in terms of the average complex permeability (8.59), frequency, and external field strength:

$$\underline{S} = P_0 + jQ_0 = H_0^2 j\omega\mu_0(\mu_1 - j\mu_2) \tag{8.65}$$

Total core loss (due to reversal of magnetization). Let us consider the simple case of linear medium. The hysteresis loss can be found from the area of the hysteresis loop in the form of an ellipse (*BH* curve) that represents the energy dissipated as heat during each hysteresis cycle. The permeability of the substance proper (see Sec. 3.4) is given as a complex quantity:

$$\mu = \mu' - j\mu'' = \mu e^{-j\delta} \tag{8.66}$$

If we replace the permeability of a substance by its complex permeability, Eq. (8.59) will give, as before, the average (effective) complex permeability of the plate. Eqs. (8.64) and (8.65) express the total core losses as a combination of eddy current losses and hysteresis losses. The division of the total core losses into the two types of loss presents a far from simple problem.

Depth of penetration. From the basic solution (8.39) it is seen that as plane waves propagate inside a conducting medium, the amplitude of each component decays exponentially. As the wave penetrates the metal for a distance

$$x_0 = 1/\alpha = \mathrm{Re}\zeta/\sigma = 1/\sqrt{\omega\mu\mu_0\sigma/2} \tag{8.67}$$

the wave amplitude decreases by a factor of $e^{\alpha x_0} = e$. This distance is conditionally called *the depth of penetration*, or *skin depth*. In reality, a field certainly penetrates a medium to a greater depth as the wave gradually decreases in amplitude.

Phase velocity v_{ph}. For plane waves moving in a conducting medium, v_{ph} can be defined as the velocity with which the crest (a maximum of the field strength) of a sinusoidal wave advances over the surface in the absence of a wave coming from the opposite direction. The wavelength λ (8.41) is defined as the distance between the two neighboring crests, one following the other. If a wave moves in a medium described by real parameters ($\alpha = \beta = \sqrt{\omega\mu\mu_0\sigma/2}$), in a distance equal to $x = \lambda = 2\pi/\beta$, the amplitude of the wave will decrease by a factor of $e^{2\pi}$, i.e. to approximately 0.001 8 of its initial value.

Obviously, a body can be considered completely nontransparent to an electromagnetic field whenever the body dimensions are comparable with the wavelength. By definition, however, bodies are considered nontransparent if the

amplitude of the wave that must cover a path in excess of a threefold skin depth decreases to 5 %. If the radius of curvature of a surface exceeds 2 to 3 times the skin depth, the surface may be assumed flat in approximate calculations and the waves that propagate into a conducting medium may be regarded as plane waves.

Cylindrical shield. In Fig. 8.10 is shown a hollow metallic cylinder of infinite length. It must protect (shield, or screen) the internal region from an external varying magnetic field whose strength $\dot{\mathbf{H}}_0$ is parallel to the cylinder axis. The cylinder radius is by far larger than the wall thickness, $r_i \gg r_e - r_i = a$. The conductivity and permeability of walls are σ and μ respectively.

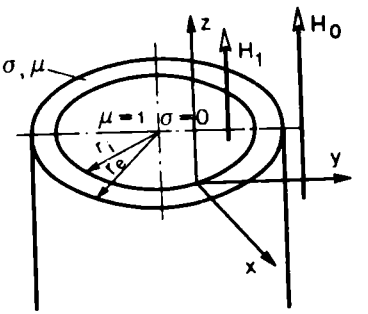

Fig. 8.10

It is easy to calculate approximately the shielding effect of such a magnetic shell if we assume that plane waves move inside the shell walls, so that

$$\dot{H} = A_1 e^{-\gamma x} + A_2 e^{\gamma x} \qquad (8.68)$$

The condition is that we count off the coordinate x from the internal surface of the cavity.

The constants A_1 and A_2 can be determined from the two known values of the electric or the magnetic field at certain values of the coordinate. Thus, knowing the internal and the external magnetic field, $\dot{H}_1$ and $\dot{H}_0$, on the inside and outside of the shell, we find that

$$\dot{H}_1 = A_1 + A_2, \dot{H}_0 = A_1 e^{-\gamma a} + A_2 e^{\gamma a} \qquad (8.69)$$

The conditions on the two surfaces are seen to be different, whereas they were identical for the transformer core lamination considered above. But asymmetry, $\dot{H}_1 \neq \dot{H}_0$, is not entirely the case in point. What is more important is that the statement of the problem specifies only $\dot{H}_0$ and it is the quantity $\dot{H}_1$ that we have to determine. The shielding (screening) constant, i.e. the shielding effect reached, is given by

$$k_{sh} = H_1/H_0 \qquad (8.70)$$

So, to evaluate the shielding effect, we must determine H_1, given the value of H_0.

But H_1 is uniquely related to the E_1 field on the inside of the shield, first, by the equation

$$\dot{E}_1 = \zeta(A_1 - A_2) \qquad (8.71)$$

and, second, by the law of electromagnetic induction

$$2\pi r_i \dot{E}_1 = -j\omega\pi r_i^2 \mu_0 \dot{H}_1 \qquad (8.72)$$

The latter equations are deduced on the assumption that $\dot{H}_1 = \dot{H}_{1z}$ and $\dot{E}_1 = \dot{E}_{1y}$ (see Fig. 8.10).

Comparing (8.71), (8.72) and the first equality in (8.69), we obtain

$$A_2 = A_1(2\zeta + j\omega r_i\mu_0)/(2\zeta - j\omega r_i\mu_0) \tag{8.73}$$

Substituting (8.73) into (8.69) gives after simple manipulations:

$$\dot{H}_0/\dot{H}_1 = \cosh\gamma a + r_i\zeta\sinh\gamma a/2 \tag{8.74}$$

The above calculation is only valid when the wavelength of a varying magnetic field in the medium surrounding the shell and filling its cavity by far exceeds the shield diameter. Really, it is only under these conditions that the sinusoidally varying magnetic field is simultaneously the same at all the points on the external shield surface ($\dot{H} = \dot{H}_0$ at $r = r_e$). The same concerns the internal field in the cavity ($\dot{H} = \dot{H}_1$ at $r \leqslant r_i$).

8.3. The Skin Effect in a Coaxial Cable

Strong skin effect. Consider the case when the depth of penetration of a field into the conductor and sheath of a cable (Fig. 8.11) is small in comparison with the conductor radius and sheath thickness. The field both in the conductor and in the sheath is really a plane wave propagating to the center in the conductor and toward the periphery in the sheath (Fig. 8.12).

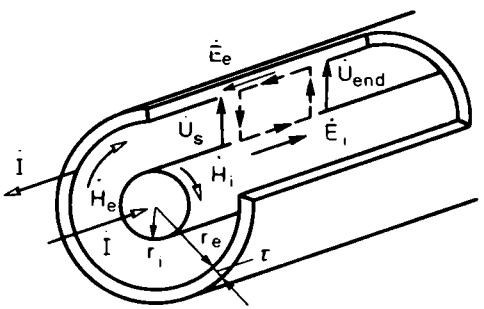

Fig. 8.11

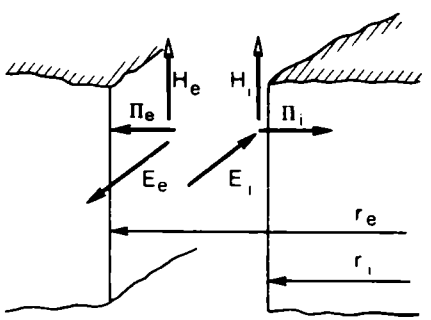

Fig. 8.12

Let the (inner) conductor carry current $\dot{I}$ into the paper as shown in Fig. 8.11. The magnetic field on the conductor surface

$$\dot{H}_i = \dot{I}/2\pi r_i \tag{8.75}$$

is then in the clockwise sense (this is the positive H direction at positive current i). The corresponding electric field is given by

$$\dot{E}_i = \zeta_i\dot{H}_i \tag{8.76a}$$

where ζ_i is the characteristic impedance of the conductor material.

The positive vector $\dot{\mathbf{E}}_i$ is in the direction of current through the conductor. With the pair of vectors $\dot{\mathbf{E}}_i$ and $\dot{\mathbf{H}}_i$ directed so as shown in Fig. 8.12, the wave propagates

into the conductor; the Poynting vector has the same direction as the wave.

In the sheath the positive current is opposite in direction to the positive vector of the electric field (see Fig. 8.12). The pair of vectors $\dot{\mathbf{E}}_e$ and $\dot{\mathbf{H}}_e$ on the outside of the conductor (sheath) are related to each other by the equation

$$\dot{E}_e = \zeta_e \dot{H}_e \tag{8.76b}$$

where ζ_e is the characteristic impedance of the sheath material; the Poynting vector is in the direction of wave propagation — the vector points to the periphery. The magnetic field on the inside of the sheath is

$$\dot{H}_e = \dot{I}/2\pi r_e \tag{8.77}$$

In the case of a strongly pronounced skin effect, the frequency must sufficiently be high as follows from the condition of the depth of penetration:

$$x_0 = 1/\sqrt{\omega\mu\mu_0\sigma/2} \ll \begin{cases} r_i \\ \tau \end{cases} \tag{8.78}$$

where τ is the sheath thickness. But the frequency must be such that the wavelength in the medium insulating the conductor from the sheath can be great as against the distance between the sheath and conductor:

$$r_e - r_i \ll \lambda_\varepsilon = vT = (c/\sqrt{\varepsilon})(2\pi/\omega) \tag{8.79}$$

In Eq. (8.79) c is light velocity and ε is the permittivity of the insulating medium; it is assumed that this medium is ideal ($\varepsilon'' = 0$ and $\sigma = 0$), and its permeability is $\mu = 1$. If the condition (8.79) were not met, we would have to consider the wave propagation inside the insulating medium in the radial direction. It would then be impossible to determine the magnetic field strength by the circuital law expressed by (8.75) and (8.77). Indeed, in the space between the sheath and conductor, an appreciable displacement current could flow in the longitudinal direction.

To transfer energy over the cable, a voltage must be applied between the conductor and sheath; a radial component of the electric field strength corresponds to this voltage. This component together with the magnetic field strength produces an axial component of the flow of power. As with the dc cable (see Sec. 7.2 and Fig. 7.7), here the axial components of the electric field are only responsible for the branching of the flow of power transmitted over the cable into the sheath and conductor.

Cable impedance. To define the complex quantity Z_0 which is the impedance of the cable per unit length, we need to find the voltage drop per unit length and then to divide it by the current in the cable.

Let $\dot{U}_s$ be the voltage between the sheath and conductor at the start of the cable portion, and $\dot{U}_{end}$ be the voltage at the end of this portion (see Fig. 8.11). We then express the above statement in the form

$$(\dot{U}_s - \dot{U}_{end})/l = \dot{I}Z_0 = \dot{I}(R_0 + jX_0) \tag{8.80}$$

The voltage in each particular cross section of the cable can be represented as the potential difference between the conductor and sheath or as the integral of the electric field along the radius:

$$\dot{U} = \dot{\varphi}_i - \dot{\varphi}_e = \int_{r_i}^{r_e} \dot{E}_r dr \tag{8.81}$$

Since we know the axial components of the field strength on the surface of the conductor and of the sheath, $\dot{E}_i$ and $\dot{E}_e$, the sought-for voltage drop, $\dot{U}_s - \dot{U}_{end}$, may be found by applying Maxwell's second equation to a rectangular loop shown by the dashed line in Fig. 8.11. The rectangle is formed by two intercepts of the radii and two parallel intercepts running along the external surface of the conductor and internal surface of the sheath. The direction of tracing round the loop and the corresponding positive direction of the magnetic flux are in agreement with the right-hand screw rule:

$$\oint \dot{\mathbf{E}} d\mathbf{l} = (\dot{E}_i + \dot{E}_e)l + \dot{U}_{end} - \dot{U}_s = -j\omega \int \dot{\mathbf{B}} d\mathbf{S} = -j\omega\dot{\Phi} \tag{8.82a}$$

or

$$\dot{U}_s - \dot{U}_{end} = (\dot{E}_i + \dot{E}_e)l + j\omega\dot{\Phi} \tag{8.82b}$$

The magnetic flux linking the loop is the flux specific to "air" inductance:

$$\dot{\Phi} = \dot{I}L_a = \dot{I}l(\mu_0/2\pi)\ln(r_e/r_i)$$

The electric field strength generally contains both the potential component $\dot{\mathbf{E}}_\varphi = -\operatorname{grad}\dot{\varphi}$ and circulation component $\dot{\mathbf{E}}_A = -j\omega\dot{\mathbf{A}}$. Separating out these components in formula (8.82a), we obtain

$$(\dot{E}_{\varphi i} + \dot{E}_{\varphi e})l - j\omega(\dot{A}_i + \dot{A}_e)l + \dot{U}_{end} - \dot{U}_s = -j\omega\dot{\Phi} \tag{8.82c}$$

But $(\dot{A}_i + \dot{A}_e)l$ is the integral of the vector potential. By definition, this integral is equal to the magnetic flux linking the loop of integration. On the strength of this analysis, we can cancel out the appropriate terms in formula (8.82c) and write it in the form

$$(\dot{E}_{\varphi i} + \dot{E}_{\varphi e})l = \dot{U}_s - \dot{U}_{end} \tag{8.82d}$$

Equation (8.82d) quite vividly shows that *the voltage drop is entirely defined as the drop of potential* along the conductors. This is indeed so since under the given conditions the voltage is the difference of potentials between the conductor and sheath in the given cross section (z-constant). The voltage drop (8.82d) is therefore *the difference of potential differences*.

Where the skin effect is very strong, $\dot{E}_i$ and $\dot{E}_e$ are described by formulas (8.76) and the magnetic fields in these formulas are expressed in terms of $\dot{I}$ in accordance with the circuital law. After simple transformations, from (8.82b) we then have

$$(\dot{U}_s - \dot{U}_{end})/l = \dot{I}j\omega L_{0a} + \zeta_i\dot{I}/2\pi r_i + \zeta_e\dot{I}/2\pi r_e = \dot{I}Z_0$$

whence

$$Z_0 = R_0 + jX_0 = \zeta_i/2\pi r_i + \zeta_e/2\pi r_e + j\omega L_{0a} \tag{8.83}$$

where

$$\zeta_{i,e} = \sqrt{j\omega\mu_{i,e}\mu_0/\sigma_{i,e}} = (1+j)\sqrt{\omega\mu_{i,e}\mu_0/2\sigma_{i,e}}$$

The first two terms contain both the real part equal to resistance R_0 and the imaginary part which is the internal reactance, the last term being the inductive reactance, i.e. the opposition offered by air inductance:

$$jX_{0a} = j\omega L_{0a} = j\omega(\mu_0/2\pi)\ln(r_e/r_i)$$

It is of importance to note that the first two terms in (8.83) are inversely proportional to the perimeter rather than to the cross section as is the case with the dc resistance.

Of course, the dc resistance R_{0dc} is lower than the ac resistance R_0. This is obvious from the fact that at the given current I, the heat loss is at a minimum if the current density is distributed uniformly over the cross section.

The general case of skin effect. If the conductors are transparent or semitransparent to electromagnetic waves, i.e. if the skin depth for a cable is of the same order of magnitude as the conductor radius r_i and sheath thickness τ, one has to consider the penetration of an electromagnetic field into the conductor across its whole thickness. The differential equation subject to the solution is then more complex in form.

Equation (8.12) for an electric field

$$\nabla^2 \dot{\mathbf{E}} = j\omega\mu\mu_0\sigma\dot{\mathbf{E}} \tag{8.84}$$

takes a comparatively simple form in the cylindrical system of coordinates since $\dot{E} = \dot{E}_z$:

$$(1/r)(\partial/\partial r)(r\partial\dot{E}/\partial r) = j\omega\mu\mu_0\sigma\dot{E} \tag{8.85}$$

Here we can use the formula for the nable square of a scalar, $\nabla^2\varphi$. It should also be pointed out that $\dot{E}$ is independent of either the z coordinate or the α coordinate.

The magnetic field strength has only the αth component. Also, the operation ∇^2 is not identical to the operation carried out on a scalar. The simplest way of finding the corresponding expression in the cylindrical coordinate system is to differentiate successively in the cylindrical coordinates. Since $\dot{H} = \dot{H}_\alpha$, we have

$$\operatorname{curl}\dot{\mathbf{H}} = \mathbf{e}_z[(1/r)(\partial/\partial r)(r\dot{H})]$$

and, in accordance with (8.5) through (8.7),

$$\operatorname{curl}\operatorname{curl}\dot{\mathbf{H}} = -\mathbf{e}_\alpha(\partial/\partial r)[(1/r)(\partial/\partial r)(r\dot{H})] = -j\omega\mu\mu_0\sigma\dot{\mathbf{H}} \tag{8.86}$$

Replacing the only argument r in (8.85) and (8.86) by the variable

$$\xi = r\sqrt{-j\omega\mu\mu_0\sigma} = x\sqrt{-j} \tag{8.86a}$$

simplifies these equations to the form

$$d^2\dot{E}/d\xi^2 + (1/\xi)(d\dot{E}/d\xi) + \dot{E} = 0 \tag{8.87a}$$

$$d^2\dot{H}/d\xi^2 + (1/\xi)(d\dot{H}/d\xi) + (1 - 1/\xi^2)\dot{H} = 0 \tag{8.87b}$$

In the latter equations partial derivatives are replaced by common derivatives since both $\dot{H}$ and $\dot{E}$ are the functions of the only variable ξ.

Note once more that $\dot{E} = \dot{E}_z$ and $\dot{H} = \dot{H}_\alpha$, for which reason (8.87a) differs from (8.87b), though both represent the expression (in cylindrical coordinates) of the same equation (8.12).

In addition to Eqs. (8.87), there are three boundary conditions:

$$\dot{H} = \dot{H}_i = \dot{I}/2\pi r_i \quad \text{at} \quad r = r_i \tag{1}$$

$$\dot{H} = \dot{H}_e = \dot{I}/2\pi r_e \quad \text{at} \quad r = r_e \tag{2}$$

$$\dot{H} = 0 \quad \text{at} \quad r = r_e + \tau \tag{3}$$

We can also specify the fourth condition when $\dot{H}$ and $\dot{E}$ remain limited at $r = 0$. Conditions 1 and 4 serve for determining two integration constants in the solution for

a cable conductor, and conditions 2 and 3 for specifying two integration constants in the solution for a cable sheath.

Equations (8.87) have received the most thorough study in mathematics. They represent particular cases of Bessel's nth-order equation:

$$d^2y/dx^2 + (1/x)(dy/dx) + (1 - n^2/x^2)\, y = 0$$

Equation (8.87a) is the zero-order equation, and (8.87b) is the first-order equation. The solutions to these equations are written as a linear combination of Bessel's functions of the first kind, $J_n(\xi)$, and second kind, $Y_n(\xi)$, of the nth order.

Thus, for an electric field

$$\dot{E} = \dot{E}_z = A_0 J_0(\xi) + B_0 Y_0(\xi) \tag{8.88a}$$

and for a magnetic field

$$\dot{H} = \dot{H}_\alpha = A_1 J_1(\xi) + B_1 Y_1(\xi) \tag{8.88b}$$

where A_n and B_n stand for integration constants.

It is practically sufficient to determine either the magnetic or the electric field strength component since the other component can easily be found from one of Maxwell's equations.

In the specified conditions of the stated problem, it is most convenient to represent $\dot{H} = \dot{H}_\alpha$ in terms of $\dot{E} = \dot{E}_z$ by Maxwell's second equation which leads to the expression

$$\dot{H} = (1/j\zeta)(dE/d\xi) \tag{8.89}$$

where ζ is the wave impedance of a medium. It should be kept in mind here that $dJ_0(\xi)/d\xi = -J_1(\xi)$.

For a cable conductor ($r \leqslant r_i$), we should set $B_1 = B_2 = 0$ in (8.88) according to the fourth condition since as the argument ξ approaches zero, the second-kind functions $Y_0(0)$ and $Y_1(0)$ go to infinity. Next, applying the first condition, we find A_1 and then A_0 related to A_1 by (8.89). The resultant of this operation is then

$$\begin{aligned} \dot{E} &= -j\zeta \dot{H}_i J_0(\xi)/J_1(\xi_i) \\ \dot{H} &= \dot{H}_i J_1(\xi)/J_1(\xi_i) \end{aligned} \tag{8.90}$$

where $\dot{H}_i = \dot{I}/2\pi r_i$ is the magnetic field on the conductor surface; and ξ_i is the argument at $r = r_i$.

The conductor impedance per unit length, equal to $\dot{E}_i/\dot{I}$ according to the definition, takes the form

$$Z_{0i} = \dot{E}_i/\dot{I} = R_0[\xi_i J_0(\xi_i)/2J_1(\xi_i)] \tag{8.91}$$

The factor $R_0 = 1/\pi r_i^2 \sigma$ is the conductor dc resistance.

Figure 8.13 presents the plots of functions $J_0(\xi)$ and $J_1(\xi)$ at $\xi = x\sqrt{-j}$.

For a coaxial cable sheath of thickness τ, field expressions (8.88) contain first-kind functions (J_0 and J_1) and second-kind functions (Y_0 and Y_1) because both remain finite when their argument is finite in magnitude and different from zero.

At large arguments, however, the absolute values of these functions become very high.

But there are linear combinations of these functions .

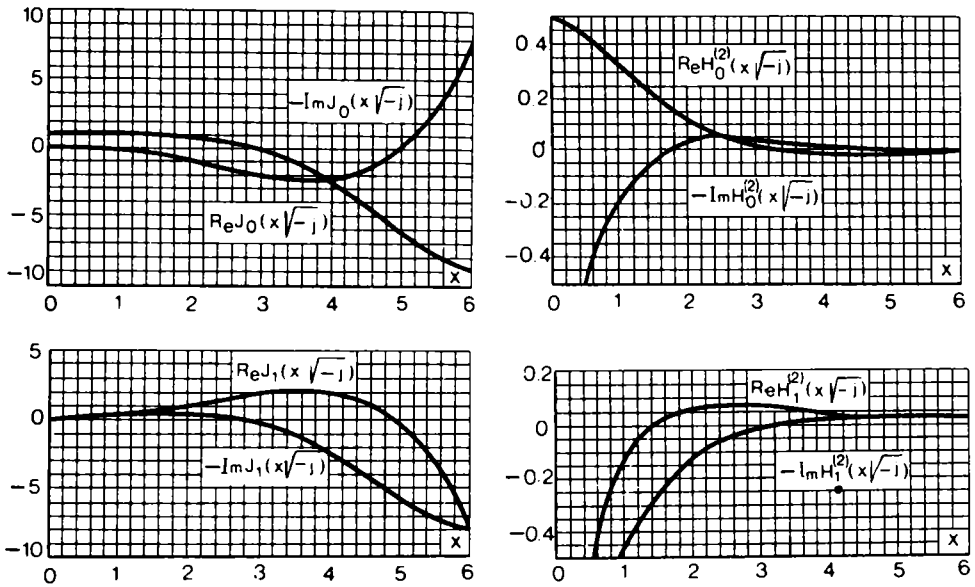

Fig. 8.13

$$H_n^{(2)} = J_n - jY_n \tag{8.92}$$

which approach zero with a rise of x at the argument $\xi = x\sqrt{-j}$:

$$\lim_{x \to \infty} H_n^{(2)}(x\sqrt{-j}) = 0$$

It is exactly these functions that correspond to physical conditions: the wave of an electromagnetic field that travels in the sheath along the radius must reduce to zero with a rise in the argument $x = r\sqrt{\omega\mu\mu_0\sigma}$. The function $H_n^{(2)}$ is known as Hunkel's function, or Bessel's function of the third kind, where n identifies the order of the function and index (2) characterizes the form of Hankel's functions. In wide use are also the functions with index (1).

The functions (8.92) express the wave going from the axis to the periphery along the radius. The plots of functions $H_0^{(2)}(\xi)$ and $H_1^{(2)}(\xi)$ are shown in Fig. 8.13.

If the sheath of a coaxial cable is nontransparent to an electromagnetic wave, the expressions for the field simplify to

$$\begin{aligned} \dot{E} &= \dot{E}_z = a_0 H_0^{(2)}(\xi) \\ \dot{H} &= \dot{H}_\alpha = a_1 H_1^{(2)}(\xi) \end{aligned} \tag{8.93}$$

The constants here are given by

$$a_0 = -j\zeta[\dot{H}_1/H_1^{(2)}(\xi_e)], \quad a_1 = \dot{H}_1/H_1^{(2)}(\xi_e)$$

where $\dot{H}_1 = \dot{I}/2\pi r_e$ is the magnetic field strength on the inside of the sheath.

The constant a_1 is directly found from the given value of $\dot{H}_1$. To determine the value of a_0, we must apply Maxwell's first equation to the expression for $\dot{H}$, con-

sidering that

$$\partial H_1^{(2)}(\xi)/\partial r = (\xi/r)[\partial H_1^{(2)}(\xi)/\partial \xi] = (\xi/r)[H_0^{(2)}(\xi) - H_1^{(2)}(\xi)/\xi]$$

According to (8.83) and (8.91), the internal impedance of a sheath nontransparent to waves is defined as

$$Z_{01} = (-j\zeta/2\pi r_e)[H_0^{(2)}(\xi_e)/H_1^{(2)}(\xi_e)] \tag{8.94}$$

If the cable sheath is "transparent", an opposite wave appears on its external surface (its expression is the same as that for the wave in the conductor). The magnitude of this wave is such that the total magnetic field strength on the sheath surface is equal to zero. But the electric field strength is different from zero.

8.4. Electromagnetic Field in Conductors of a Two-Wire Line

Large distance between conductors. If the distance $2a$ between the two wires of a transmission line is large in comparison with their radius r_0, the effect of the magnetic field of the wire on its neighbor can be neglected. This permits us to calculate the impedance of each of the two parallel round wires (considering the skin effect) as the coaxial cable conductor impedance (8.91). This impedance should also include the reactance attributed to the magnetic flux in the air.

The impedance for the two-wire line with round conductors of radius r_0 is then given by

$$Z_0 = (\xi_i/\pi r_i^2\sigma)[J_0(\xi_i)/J_1(\xi_i)] + j\omega(\mu_0/\pi)\ln(2a/r_0) \tag{8.95}$$

Short distance between conductors. In this case each conductor is found to be not only in the field of the current it carries but also in the current field of its neighbor. As a result, the electric field induced in one conductor with changes in the magnetic field of the other has an added effect on the distribution of the electric field in the first conductor. This effect is referred to as a *proximity effect*.

Unfortunately, a more or less simple integration of equations is impossible in this case.

It is common to use experimentally derived formulas for conductors of standard configuration; pertinent reference books usually give the calculated resistances and reactances of the conductors of a certain grade and construction. Note also that conductors come solid and stranded of a number of wires. They are often made of two metals such as steel and aluminum and also steel and copper. Accurate calculations of such conductors are not feasible. Before using conductors of a new cross section with a new mutual arrangement it is advisable to carry out preliminary measurements or perform tests on models.

Conductor — ground. In power transmission lines and also communication systems, of particular interest is the case where the return conductor for current is the ground (earth).

In ac power systems, this case of operation is generally the case of emergency. This is not so in dc systems. The only factor to be reckoned with is the resistance at the spots where the current dissipates into the earth from grounding conductors; along the rest of the path, the current spreads out over such a large cross section that the

resistance practically reduces to zero. For an ac line, the depth of penetration of the field into ground is not too large (due to the skin effect), so that the following relation is valid:

$$\dot{E}_g/\dot{I} = R_{0g} + jX_{0g}$$

Here the electric field $\dot{E}_g$ on the ground surface determines the resistance and reactance offered by the ground to the current flowing into it. For power lines, it may be assumed that the resistance is practically independent of ground conductivity and equal to around 0.05 ohm/km at 50 Hz. The reason is that current spreads out to a smaller depth with an increase in ground conductivity It is customary to assume the reactance of a wire-ground loop at 50 Hz equal to

$$X_{w\text{-}g} = 0.338 \ln(D/a_w)$$

where a_w is the wire radius; D is a certain reference depth at which the equivalent, return wire runs, the depth being dependent on the ground resistivity ρ. For the ground of medium humidity ($\rho = 10^2$ ohm m), the depth D is 940 m, and for stony ground ($\rho = 10^4$ ohm m) this depth is 9.4 km.

The effect of an internal magnetic field in determining the inductance. If we regard the inductance as a coefficient in the expression for reactance, it makes sense to consider the effect of an internal magnetic field only where the emf induced by this field cannot be taken negligible in comparison with the emf due to the flux between the wires or in comparison with the active component of the voltage drop in the wires. In these cases, the flow of alternating current is inevitably associated with a nonuniform distribution of the current in the wire (i.e. with the skin effect). Therefore, the effect of the internal magnetic field must be taken into account in such a manner as is done in the theory of skin effect. Where the emf caused by the internal flux is small and does not lead to a redistribution of current in the wire, the effect of the internal magnetic field may simply be neglected.

Thus, when we evaluated the self-inductance of the loop comprising two parallel wires (when $2a \gg r_0$), we assumed that

$$L_0 = (\mu_0/\pi) \ln(2a/r_0)$$

But in the literature on the subject this formula often contains an additional term that allows for the flux inside the wires proper:

$$L_0 = (\mu_0/\pi)[\ln(2a/r_0) + 0.25] \tag{8.96}$$

If this term plays a noticeable part in comparison with the first term and, besides, the inductive reactance ωL_0 cannot be taken negligible as against the resistance R_0, then it is necessary to allow for the internal inductance, proceeding from the calculation of field distribution in the wire, and, sometimes, to give due consideration for the proximity effect.

8.5. Conductors in the Slot of an Electrical Machine

Calculation for the case of a plane wave. It may be assumed that a z-directed plane wave propagates in copper or aluminum conductors ($\mu = 1$) inserted into a ferromagnetic slot (Fig. 8.14), the electric and the magnetic field of the wave being

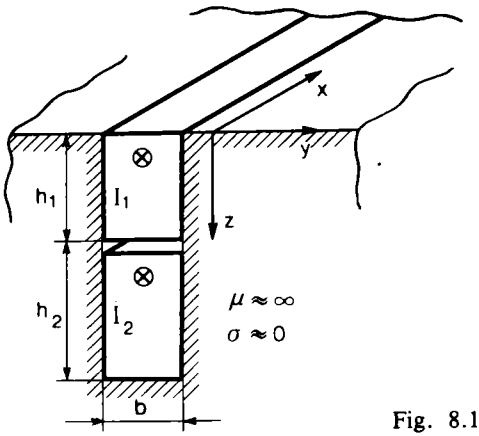

Fig. 8.14

$\dot{E} = \dot{E}_x$ and $\dot{H} = \dot{H}_y$ respectively.

This assumption is valid primarily because a ferromagnet has high permeability ($\mu \approx \infty$), the conductivity of which is zero in the x direction since the ferromagnet consists of steel laminations insulated one from another and disposed normal to the machine axis (the x axis). The magnetic field vector must then enter the slot practically at a right angle to its walls. Hence, we can state that the field lines run approximately parallel to the y axis.

For the system being examined, the general expression for $\dot{\mathbf{H}}$ and $\dot{\mathbf{E}}$, which is similar to 8.39, may be assumed to be known:

$$\dot{H} = \dot{H}_y = A_1 e^{-\gamma z} + A_2 e^{\gamma z} \tag{8.97a}$$

$$\dot{E} = \dot{E}_x = C_1 e^{-\gamma z} + C_2 e^{\gamma z} \tag{8.97b}$$

where $\gamma = \sqrt{j\omega\mu_0\sigma}$.

Of the two equations with independent integration constants A_1, A_2, C_1, and C_2 it is advantageous to have one equation derived from the other by resorting to one of Maxwell's equations. Thus, applying the first equation curl $\dot{\mathbf{H}} = \sigma\dot{\mathbf{E}}$ and noting that $\dot{H} = \dot{H}_y$ and the field depends only on the z coordinate, we find, after the substitution of $\dot{H}$ from (8.97a), that

$$\dot{E} = \dot{E}_x = \zeta(A_1 e^{-\gamma z} - A_2 e^{\gamma z}) \tag{8.98}$$

where $\zeta = \gamma/\sigma$ as before.

It now remains to determine only the constants A_1 and A_2 proceeding from the concrete statement of the problem. Let the slot contain two wires of heights h_1 and h_2 (see Fig. 8.14) that carry currents $\dot{I}_1$ and $\dot{I}_2$. Let these wires be copper conductors and hence have the same values of γ and ζ.

Solve the problem first for the lower conductor. For this, transfer the reference point of z to the top of the lower conductor. The constants A_1 and A_2 are found from boundary conditions for $z = 0$ and $z = h_2$.

At $z = h_2$ (the slot bottom), the magnetic field strength is zero:

$$\dot{H}_y = A_1 e^{-\gamma h_2} + A_2 e^{\gamma h_2} = 0 \tag{8.99}$$

Indeed, at a finite value of $\dot{I}_1 + \dot{I}_2$ the magnetic flux encircling the conductors in the open slot must be finite because the flux must close on itself through the air. But if the flux is finite and the ferromagnet permeability is very high (theoretically infinite), then the field strength in the ferromagnet must be equal to zero. Besides, the tangential component of the field strength does not undergo discontinuity (does not change abruptly) at the boundary $z = h_2$. The above analysis leads us to the conditions (8.99).

At $z = 0$ (between the conductors)

$$\dot{H} = A_1 + A_2 = \dot{I}_2/b \tag{8.100}$$

where b is the slot width.

The latter condition directly follows from (8.97a) in combination with the circuital law: integrating around the loop traced from the left to the right wall between the conductors and closed through the ferromagnet (see Fig. 8.14), we obtain

$$\oint \dot{\mathbf{H}} d\mathbf{l} = \dot{H}_y b = \dot{I}_2$$

From (8.99) and (8.100) we find that

$$A_1 = (\dot{I}_2/2b)(e^{\gamma h_2}/\sinh \gamma h_2)$$
$$A_2 = (-\dot{I}_2/2b)(e^{-\gamma h_2}/\sinh \gamma h_2)$$

Using (8.97a) and (8.98) yields

$$\dot{H} = (\dot{I}_2/b)[\sinh \gamma (h_2 - z)/\sinh \gamma h_2] \tag{8.101a}$$

and

$$\dot{E} = \zeta(\dot{I}_2/b)[\cosh \gamma (h_2 - z)/\sinh \gamma h_2] \tag{8.101b}$$

where z is counted off from the top of the lower conductor.

The above solution permits us to find readily the solution for the simplest case of a single conductor in the slot. For this it is enough to set the total depth of the slot equal to $h = h_2$, which enables us to assume that $\dot{I}_2 = \dot{I}$, $\dot{I}_1 = 0$, and $h_1 = 0$.

A similar approach is applicable for obtaining the solution for the upper conductor. Transferring the reference point for z to the top of the upper conductor of Fig. 8.14 and determining A_1 and A_2 from boundary conditions $\dot{H} = \dot{I}_2/b$ for $z = h_1$ and $\dot{H} = (\dot{I}_1 + \dot{I}_2)/b$ for $z = 0$, we get

$$\dot{H} = \frac{\dot{I}_1}{b} \frac{\sinh \gamma (h_1 - z)}{\sinh \gamma h_1} + \frac{\dot{I}_2}{b} \frac{\sinh \gamma z + \sinh \gamma (h_1 - z)}{\sinh \gamma h_1} \tag{8.102a}$$

$$\dot{E} = \frac{\zeta}{b} \left[\dot{I}_1 \frac{\cosh \gamma (h_1 - z)}{\sinh \gamma h_1} - \dot{I}_2 \frac{\cosh \gamma z - \cosh \gamma (h_1 - z)}{\sinh \gamma h_1} \right] \tag{8.102b}$$

In the system under study we have encountered all the typical features peculiar to the propagation of plane waves: the field results from two waves, one traveling opposite to the other. At the lowermost point of the second conductor, i.e. at the slot bottom, the magnetic field is zero. This means that at the slot bottom, the magnetic field vectors of the two waves moving in the opposite directions are equal and opposite in sign.

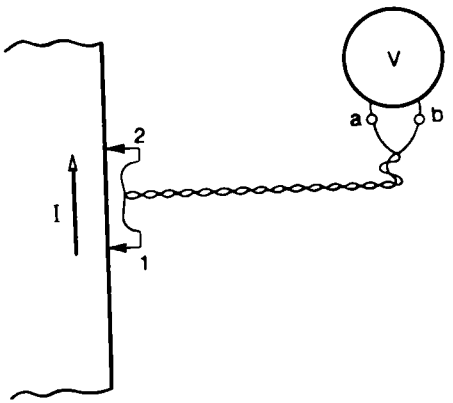

Fig. 8.15

Although the calculation is approximate in character (the assumption is that everywhere $\dot{H} = \dot{H}_y$), it gives the results which are rather close to the ones obtained in measurements. The experimental study of the $\dot{E}$-z relation can be made by measuring the voltage $\dot{E}_0 l = \dot{U}$ between two sharp electrodes pressed against the conductor surface (Fig. 8.15). The wires running from the electrodes to the voltmeter must be disposed so that the closed loop along which the integral is taken cannot be linked, where possible, by the magnetic flux. Only then can we say that

$$\oint \dot{\mathbf{E}} d\mathbf{l} = \dot{E}_0 l - \dot{U} = 0$$

where $\dot{U}$ is the voltage read from the voltmeter. Where the above precautions are not taken, the right-hand side of the equation becomes other than zero, namely, $-j\omega\dot{\Phi}$.

Energy dissipation in conductors. What is the resistance of a conductor in a slot? What amount of loss does the skin effect contribute to the total thermal losses in the conductor?

The energy lost in conductors can easily be found from the Poynting vector. Thus, for the lower conductor of Fig. 8.14, the incoming Poynting flux passes only through the upper surface of the conductor. This flux per unit length of the conductor is given by

$$P_0 + jQ_0 = \dot{E}\dot{H}^* b \tag{8.103}$$

The real part of (8.103) is just the sought-for energy irreversibly dissipated in heat which can be represented as the usual $I^2 R_0$ loss.

After the substitution of $\dot{E}$ and $\dot{H}$ from (8.101) at $z = 0$ (for the upper surface of the lower conductor), we have

$$I^2 R_0 = P_0 = b\,\mathrm{Re}(\dot{E}\dot{H}^*) = I_2^2\,\mathrm{Re}\,[(\zeta/b)\coth\gamma h_2]$$

The ratio of the real part enclosed in the brackets to the dc resistance serves as a measure of an increase in the resistance due to the skin effect:

$$k = \mathrm{Re}(\gamma h_2 \coth\gamma h_2) \tag{8.104}$$

This formula is similar to (8.59) we have met with in discussing the propagation of plane electromagnetic waves in a conducting medium.

The resistance of one conductor of height $h = h_1 = h_2$ that occupies the entire slot exceeds one-fourth the resistance of two series-connected conductors of height $h_1 = h_2 = h/2$. The resistance of these two conductors is easy to calculate by setting $\dot{I}_1 = \dot{I}_2 = \dot{I}$.

8.6. Study of the Skin Effect on Models

It is obvious that the skin effect should be reckoned with in high frequency applications. This is also the case for power engineering which has to deal with bus ducts of large cross section enclosed in thick metallic jackets. The skin effect in such systems plays a significant part. Since the calculations often prove extremely involved, recourse is generally made to modeling.

A small-scale model of an electric system. In the study of large-size circuits intended for work at comparatively low frequencies, it is convenient to investigate the field and evaluate the impedance on a model physically similar to the original but in a reduced scale.

In order that the process of penetration of an electromagnetic field into the metal of the model should remain the same as in the system under study, it is necessary to keep invariable the product (Zenneck's parameter)

$$a^2 \omega \mu \mu_0 \sigma = \text{constant} \tag{8.105}$$

where a is one of the linear parameters of the system.

The argument of the functions characterizing the skin effect, i.e. ξ or γa, then remains invariable. The form of the function itself is determined by the shape of the cross section.

But in any of the expressions for impedance, a factor

$$\zeta/a = (\sqrt{j\omega\mu\mu_0/\sigma})/a \tag{8.106}$$

stands ahead of a certain function of ξ.

For example, the impedance of a conductor is

$$Z_0 = (\zeta/a)\, Z_0' = (\zeta/a)\, F(\xi) \tag{8.107}$$

Therefore, keeping the function $F(\xi)$ invariant, i.e. following the condition (8.105), we find that the model impedance Z_0' is different from the impedance Z_0 of the original.

The inadequacy of condition (8.105) shows up quite vividly in the example of the cylindrical shield specified by (8.74). Shielding depends not only on the quantity $\gamma a = a\sqrt{j\omega\mu\mu_0\sigma}$, whose square is equal in magnitude to Zenneck's parameter, but also on the wave impedance of the shield material, $\zeta = \sqrt{j\omega\mu\mu_0/\sigma}$. The above analysis thus reveals the difficulties met with in modeling and calculations made on the basis of model data.

There are no such difficulties in case of the magnetic skin effect, where the entire process occurs within the region of a conducting core (see Sec. 8.2), and also in case of the skin effect in conductors placed in the slots of an electrical machine (see Sec. 8.5). The conditions specific to the above cases allow modeling transients too (see Fig. 8.4, for example). For the purpose formula (8.105) can be useful after

replacing ω by τ^{-1}, where τ is a certain time constant characteristic of the transient in question. For example, the modified formula

$$a\sqrt{\mu\mu_0\sigma/\tau} = a_1\sqrt{\mu\mu_0\sigma/\tau_1} = \text{constant} \tag{8.108}$$

shows that the time scale for the model (identified with subscript 1) has been changed. The real time t for the original is equal to the estimated time t_1 for the model multiplied by the scale factor

$$k_t = \tau/\tau_1 = (a/a_1)^2 \tag{8.109}$$

if the conductivity and permeability are invariable.

Analog. The skin effect in parallel wires lends itself to simulation by an electric analog. The processes occurring in the analog are quite different from those in the original, but they are definable in terms of functional dependences proportional to the dependences valid for the original. This simulation often proves very informative.

A network of two-ports as a model. Where the skin effect is amenable to the analysis by considering the simple propagation of a plane wave, as is the case for transformer core laminations, it is possible to use an iterated network of two-ports (as a model of a long transmission line). The model for nonlinear ferromagnetic plates represents a circuit of two-ports with nonlinear inductances. Such a circuit model permits studying the process of propagation of a plane wave in a nonlinear conducting medium. Fig. 8.16 illustrates a circuit model for the study of the skin effect in a lamination. This model enables us to analyze a transient after establishing the stationary magnetic field $H_0 \approx i'$ on both sides of the model; investigate the steady-state processes at the given external sinusoidally varying field on connecting the model to the current source $\dot{I}$; and study the steady processes peculiar to the sinusoidally varying magnetic flux on connecting the model to the voltage source $\dot{U}$.

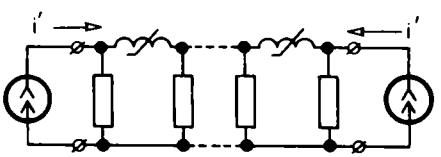

Fig. 8.16

8.7. Additional Remarks

In this chapter we have described most important technical problems. Although this chapter is comparatively large, its space is hardly sufficient to present fully the theory of a varying field in conducting media. We have not touched upon the whole class of problems dealt with in theory, largely because the methods of mathematical analysis are not always suitable for ordinary technical calculations, too cumbersome, or even poorly developed. This has forced us to consider even the classical problem on the loss in transformer core laminations only to an initial approximation. In reality, a ferromagnetic material of cores exhibits a nonlinear dependence $B(H)$, i.e. anisotropy that appears as a nonuniform permeability in different directions. A fer-

romagnet has a number of other peculiar properties which show up most vividly with fast changes in magnetization, 10^{-6} s to 10^{-9} s.

The calculations aimed at estimating the skin effect with due regard for the nonlinear properties of a material present a complicated problem, the more so as upper harmonics appear in the material. Indeed, in transformer laminations linked by a sinusoidal magnetic flux, an external nonsine field $H(t)$ is set up because the current is nonsinusoidal; if the external field (current) is sinusoidal, the magnetic induction proves nonsinusoidal. It is of interest to note that the case which is practically amenable to calculations is the one involving extremely large nonlinearity, where the hysteresis loop takes the form of a rectangle; the calculation procedure becomes somewhat analogous to that described below which takes account of the domain structure.

Some problems are solvable by the method suggested by F. E. Newmann. The method replaces the induction-magnetic field relation by the dependence of permeability on the coordinate. It offers the possibility of evaluating such important parameters as the internal impedance of ferromagnetic conductors and average permeability of cores, but the method excludes from the consideration upper harmonics that appear in ferromagnets.

The static hysteresis loop of modern electrical-sheet steel is made very narrow by virtue of the persistent effort of technologists; the loop area is very small, i.e. the energy lost in a hysteresis cycle is very low, 0.4 W/kg at 50 Hz in good steel grades. But the share of eddy current loss has become rather substantial; this loss is noticeably higher than the value calculated by the classical method for materials of given conductivity and permeability. Added losses can partially be attributed to a higher imaginary part of permeability with rising frequency (the effect is known as magnetic viscosity) and partially to the microscopic nonuniformity of magnetization.

The seemingly low iron loss has on the whole a significant bearing on the economical effectiveness. The thing is that modern industry produces annually millions of tons of electrical steel. Also, a power line, for example, generally requires several iron-core transformers for the transmission of power to consumers. Moreover, it is largely the permissible density of energy dissipated in steel and copper that determines the dimensions of huge transformers and generators installed at electric power stations; an increase in specific iron loss calls for larger machines and thus larger rooms at power stations.

Eddy-current loss increase due to the domain structure. Consider the simplest, though instructive case, which helps understand why the eddy current loss can be larger than the value obtained from classical equations (see Sec. 8.2).

From the viewpoint of the theory of ferromagnetism, a demagnetized plate, where $B = B_{av} = M_{av} = 0$, consists of a number of oppositely magnetized regions (domains) as shown in Fig. 8.17. In the plate of thickness $2a = 4b$, the outer layers of thickness b have M approaching saturation with the **M** vector pointing downward, whereas the medium layer of thickness $2b$ has the same magnetization directed upward. The average magnetization is

$$M_{av} = M_s(2b - 2b) = 0$$

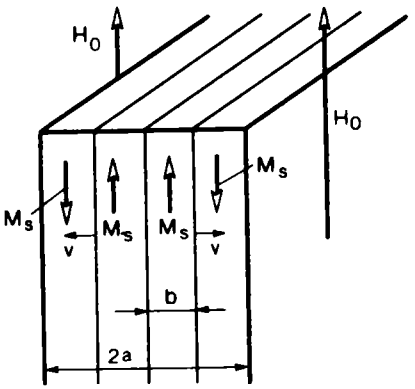

Fig. 8.17

whereas the saturation magnetization is very high, $\mu_0 M_s \approx 2\text{T}$.

Notice that at such a high magnetization, the magnetic induction (magnetic flux density) in each of the regions practically remains equal to $\mu_0 M_s$ despite the presence of an additional field strength H_0, even if this field could amount to hundreds of amperes per centimeter. We thus can always assume that in a good ferromagnet the saturation induction is equal to

$$B_s = \mu_0(M_s + H_0) \approx \mu_0 M_s$$

if, of course, the H_0 field does not reach thousands and tens of thousands of amperes per centimeter.

The process of magnetization of the plate being considered may be thought of as the process of displacement of boundaries (walls) between the domains. Thus, if an externally applied field H_0 causes the domain walls to move in opposite directions at a speed v, the mean magnetic induction $\bar{B}$ grows:

$$d\bar{B}/dt = 2v\mu_0 M_s/a \tag{8.110}$$

Really, as a wall shifts for a distance $ds = vdt$, the magnetization in the region changes from $-M_s$ to $+M_s$, i.e. by $2M_s$. But the change in induction should be related to the plate half-thickness, a. Assuming that the mean value of induction changes by the law $\bar{B}_m \cos\omega t$, we find that

$$2v\mu_0 M_s/a = -\omega \bar{B}_m \sin\omega t$$

whence

$$v = -V_m \sin\omega t \quad \text{at} \quad V_m = \omega a \bar{B}_m / 2\mu_0 M_s$$

A maximum displacement of the domain wall at such a speed is then

$$s_m = V_m/\omega = a\bar{B}_m/2\mu_0 M_s \tag{8.111}$$

Obviously, at comparatively low values of $\bar{B}$, about 0.1 T (1000 G), the relative displacement of the wall is small, $s_m/a = 0.025$, and so the width of all domains may be taken constant.

The wall motion causes an electric field in the outer regions:

$$E(t) = 2\mu_0 M_s v \tag{8.112}$$

It is easy to calculate the amount of energy that must be lost as Joule's heat in a unit volume:

$$P_0 = E^2\sigma \cdot 2b/2a = a^2\omega^2\bar{B}^2\sigma/2 \tag{8.113}$$

In the latter expression, $\bar{B}$ is the effective mean induction over the cross section, and E is the effective field being induced.

We can now compare the above result with the one obtained for eddy current losses by the method described in Sec. 8.2. At low frequencies, when $|\gamma a| \ll 1$, the expansion of $\tanh \gamma a$ into a series can evidence that the losses found from (8.113) are 1.5 times as high as the losses calculated by (8.64).

Magnetic field in the presence of superconductors. These materials are already in commercial use. They will doubtlessly play an increasingly important role in the near future. Unfortunately, we cannot, in the space available, consider the behavior of superconductors in a magnetic field and their electromagnetic properties. We refer the reader to the literature available on the subject.

Chapter 9 Propagation and Reflection of Electromagnetic Waves

The present book briefly discusses only the basic concepts of the theory of wave propagation and reflection that do not require complex mathematical calculations. A large number of monographs and textbooks are available on the subject of wave propagation, reflection, and refraction.

9.1. Plane Wave Propagation

Consider simple cases of propagation of electromagnetic plane waves. Assume a wave propagates in an ideal dielectric, $\underline{\varepsilon} = \varepsilon$; its electric vector has only the xth component and depends only on the space z coordinate, i.e. $\dot{E} = \dot{E}_x(z)$. As was shown in Sec. 8.1 [see (8.10)], the electric field vector $\dot{E} = \dot{E}_x$ then satisfies the equation

$$d^2\dot{E}/dz^2 = (j\omega/v)^2\dot{E} = \gamma^2\dot{E} \tag{9.1}$$

where $v = 1/\sqrt{\varepsilon\varepsilon_0\mu\mu_0}$ is the wave velocity, and $\gamma = j\omega\sqrt{\mu\mu_0\varepsilon\varepsilon_0} = j\beta$ is the propagation constant.

Differential equation (9.1) is the wave equation (see Ch. 8), but its solution is analogous to the solution of diffusion equation (8.38a):

$$\dot{E} = A_+e^{-\gamma z} + A_-e^{\gamma z} = \dot{E}_+ + \dot{E}_- \tag{9.2}$$

It differs from the solution of (8.38a) only in that the factor γ in the exponent is purely imaginary when ε and μ are real.

The constants A_+ and A_- are specified by boundary conditions. The transition from the complex representation to instantaneous values is accomplished in a common way:

$$E(t, z) = E_{+,m}\sin(\omega t - \beta z + \psi_+) + E_{-,m}\sin(\omega t + \beta z + \psi_-) \tag{9.3}$$

If in (9.2) $\dot{E}$ represents the effective value of the complex number, then

$$E_{+,m} = |A_+|\sqrt{2} \text{ and } E_{-,m} = |A_-|\sqrt{2}$$

The factor $\sqrt{2}$ can be omitted if $\dot{E}$ represents the amplitude value of the complex number, which is often implied to be the case even if the quantity in question has no subscript, such as m.

Expressions (9.2) and (9.3) describe a wave in an ideal dielectric. This is a real wave, and expression (9.3) fully corresponds to the definition of a wave (8.28): a disturbance that is a function of the difference between the coordinate and the product of time and speed, namely,

$$f(t, R) = f(R - vt) \tag{9.4}$$

The arguments of the two terms in (9.3) reduce exactly to the form of (9.4):

$$\beta[(\omega/\beta)t \mp z] + \psi_{+,-} \tag{9.5}$$

It is thus obvious that the first term in (9.3) represents a wave traveling at a velocity $v = \omega/\beta$ in the positive z direction (forward wave), and the second term represents a wave coming from the opposite direction at the same velocity (backward wave). Here,

$$v = \omega/\beta = c/\sqrt{\varepsilon\mu} \tag{9.6}$$

where c is the velocity of propagation of the wave in vacuum.

The wave referred to is a plane transverse-polarized wave. It varies only in the direction of propagation along the single cartesian coordinate and has the only spatial component normal to the direction of propagation and independent of the space coordinate.

Knowing the expression for one of the vectors of an electromagnetic field, we can define the other vector by applying one of Maxwell's equations to the expression for the first vector. In our case, we use the second equation, which gives

$$\dot{\mathbf{H}} = \text{curl}\,\dot{\mathbf{E}}/-j\omega\mu\mu_0 = (1/-j\omega\mu\mu_0)(d\dot{E}/dz)\mathbf{e}_y \tag{9.7}$$

since $\dot{E} = \dot{E}_x(z)$. Differentiating (9.2) yields

$$\dot{H} = \dot{H}_y = \dot{H}_+ + \dot{H}_- = (A_+ e^{-\gamma z} - A_- e^{\gamma z})/\zeta \tag{9.8}$$

where

$$\zeta = j\omega\mu\mu_0/\gamma = \sqrt{\mu\mu_0/\varepsilon\varepsilon_0} = 120\pi\sqrt{\mu/\varepsilon}\text{, ohms} \tag{9.9}$$

For vacuum (air), $\zeta = \zeta_0 = 120\pi = 377$ ohms. As mentioned earlier, the parameter ζ is the wave impedance of a medium, defined as the ratio between the vectors $\dot{E}$ and $\dot{H}$ of a forward-traveling or backward-traveling wave.

For a medium described by complex number $\underline{\mu} = \mu' - j\mu'' = \mu \angle -\delta_\mu$ and $\underline{\varepsilon} = \varepsilon' - j\varepsilon'' = \varepsilon \angle -\delta_\varepsilon$, the propagation constant γ and wave impedance ζ are complex numbers too:

$$\gamma = \alpha + j\beta = j\omega\sqrt{\mu_0\varepsilon_0\underline{\mu}\,\underline{\varepsilon}} = (\omega/c)\sqrt{\mu\varepsilon} \angle (\pi - \delta_\varepsilon - \delta_\mu)/2 \tag{9.10a}$$

$$\zeta = 120\pi\sqrt{\underline{\mu}/\underline{\varepsilon}} = 120\pi\sqrt{\mu/\varepsilon} \angle (\delta_\varepsilon - \delta_\mu)/2 \tag{9.10b}$$

The real and imaginary parts of γ characterize attenuation and the velocity of propagation, respectively.

Where γ that defines a wave is a complex quantity, the wave decays exponentially. Thus, for a plane wave traveling in the z direction (in the absence of a backward wave),

$$\dot{E} = \dot{E}_0 e^{-\gamma z} = \dot{E}_0 e^{-\alpha z} e^{-j\beta z} \tag{9.11}$$

or

$$E(t, z) = E_0 e^{-\alpha z} \sin(\omega t - \beta z + \psi_0)$$

where E_0 is the amplitude and ψ_0 is the phase of a wave at $z = 0$ and $t = 0$. For the wave in question, the magnetic field is given by

$$\dot{H} = \dot{H}_0 e^{-\gamma z} = \dot{H}_0 e^{-\alpha z} e^{-j\beta z} \tag{9.12}$$

and

$$\dot{H}_0 = \dot{E}_0 / \zeta$$

The magnetic field lags in phase the electric field by an angle $\varphi = (\delta_\varepsilon - \delta_\mu)/2$, as shown in Fig. 9.1 for $\varphi > 0$; the corresponding spatial shift is $\Delta z = \varphi/\beta$. The sketch is drawn not to scale, the aim being merely to stress out the shift in phase. With such a phase difference, the amplitudes must in reality decay appreciably faster. It is instructive to compare this wave pattern with Maxwell's sketch shown in Fig. 2.5 for the lossless case.

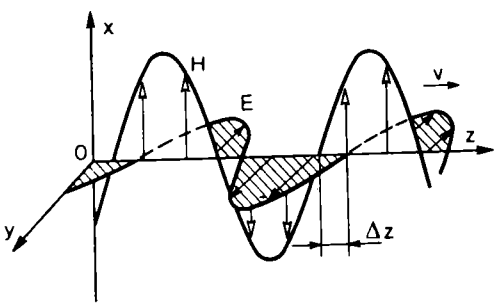

Fig. 9.1

Equations (9.10), (9.11) and the solutions found from them are also valid for a medium defined by an equivalent permittivity ε_{eq} whose imaginary part is attributable to conductivity. When a wave impinges upon a second medium, the wave will partially bounce back and partially propagate into the second medium. The wave pattern then changes substantially.

Reflection of waves from a flat boundary between two media (normal incidence). In the general case, the solution to wave equations for a given electromagnetic wave incident on a body that disturbs the homogeneity of a medium involves appreciable difficulties, though from the physical viewpoint it is a very simple matter to formulate the problem: it reduces to the solution of the wave equation $\nabla^2 \dot{\mathbf{F}} = \gamma^2 \dot{\mathbf{F}}$, with the boundary conditions for vectors $\dot{\mathbf{E}}$, $\dot{\mathbf{D}}$, $\dot{\mathbf{H}}$, and $\dot{\mathbf{B}}$ being satisfied. These problems are dealt with in radio engineering, where the area of study covers propagation of electromagnetic waves over the surface of the earth or sea, reflection of waves from

conducting layers of the atmosphere (differently ionized layers of the ionosphere), interaction of incident waves with receiving antennae, etc.

We shall restrict our analysis to the simple case of a linearly polarized plane wave incident at a right angle onto a flat interface between two isotropic media. The assumption is that the incident, reflected, and transmitted waves are equally polarized[1]. If this were not the case, it would be impossible to obtain the required solution for normal incidence, i.e. to satisfy wave equations and boundary conditions for two media.

Let us choose a coordinate system from Fig. 9.2. The parameters of the first medium ($z < 0$) are ζ_1 and γ_1 at ε_1 and μ_1; the parameters of the second medium ($z > 0$) are ζ_2 and γ_2 at ε_2 and μ_2. The incident wave advances in the positive z direction. The x and y axes are so chosen that $\dot{E} = \dot{E}_x$ and $\dot{H} = \dot{H}_y$.

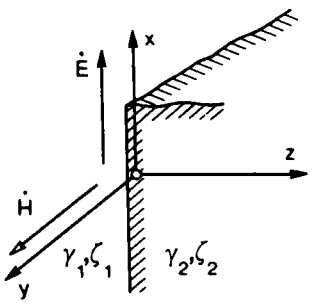

Fig. 9.2

The boundary conditions on field vectors can be written as

$$\dot{E}_1 = \dot{E}_2,\ \dot{H}_1 = \dot{H}_2 \text{ at } z = 0 \tag{9.13}$$

These conditions specify the known requirement for continuity of the tangential components of field vectors.

Under the conditions of the problem, the incident and the reflected wave propagate in the first medium:

$$\dot{E}_i = \dot{E}_{i0} e^{-\gamma_1 z},\ \dot{E}_r = \dot{E}_{r,0} e^{\gamma_1 z} \tag{9.14a}$$

$$\dot{H}_i = \dot{E}_i/\zeta_1,\ \dot{H}_r = -\dot{E}_r/\zeta_1 \tag{9.14b}$$

Referring to (9.2) and (9.8), we can state that

$$\dot{E}_1 = \dot{E}_i + \dot{E}_r \tag{9.15a}$$

$$\dot{H}_1 = \dot{H}_i + \dot{H}_r = (\dot{E}_i - \dot{E}_r)/\zeta_1 \tag{9.15b}$$

In the second infinite medium, only the transmitted wave is present, which travels along the positive z direction:

$$\dot{E}_2 = \dot{E}_{tr} = \dot{E}_{tr0} e^{-\gamma_2 z} \tag{9.16a}$$

[1] In a plane-polarized wave, the field vectors remain lying in any one of the planes over the entire extent of the wave. For example, **E**, remains parallel to x and, correspondingly, **H** parallel to y.

$$\dot{H}_2 = \dot{H}_{tr} = \dot{E}_{tr}/\zeta_2 \tag{9.16b}$$

Substituting the vectors of the incident, reflected, and transmitted waves into (9.13) at $z = 0$ and taking into account (9.15), we obtain the set of equations:

$$\dot{E}_i + \dot{E}_r = \dot{E}_{tr} \tag{9.17a}$$

$$\dot{H}_i + \dot{H}_r = \dot{H}_{tr} = (\dot{E}_i - \dot{E}_r)/\zeta_1 = E_{tr}/\zeta_2 \tag{9.17b}$$

Eliminating from these equations the vector of the transmitted wave, we find that at $z = 0$

$$\dot{E}_r = \Gamma \dot{E}_i \tag{9.18}$$

Here Γ is a reflection coefficient, given by

$$\Gamma = (\zeta_2 - \zeta_1)/(\zeta_2 + \zeta_1) \tag{9.19}$$

Since $\dot{H}_r = -\dot{E}_r/\zeta_1$ at $z = 0$ we have

$$\dot{H}_r = -\Gamma \dot{H}_i \tag{9.20}$$

Using the above expressions and recalling (9.13), we can easily obtain the vectors of the transmitted wave at $z = 0$:

$$\dot{E}_2 = (1 + \Gamma)\dot{E}_i \tag{9.21}$$

$$\dot{H}_2 = (1 - \Gamma)\dot{H}_i \tag{9.22}$$

Consider how a wave goes from the air (vacuum) into copper at $f = 50$ MHz. According to (9.9), $\zeta_1 = 120\pi = 377$ ohms and ζ_2 is the complex quantity dependent on frequency. At 50 MHz, the wave impedance of copper is

$$\zeta_2 = 2.6 \times 10^{-3} \angle 45^\circ, \text{ ohms}$$

As the frequency decreases, ζ_2 decreases accordingly.

Since $|\zeta_2/\zeta_1|$ is very small, Γ may be taken equal to about -1. The wave is fully reflected and the sign of the electric vector is changed:

$$\dot{E}_{r0} = -\dot{E}_{i0}$$

The electric field on the surface of copper is thus close to zero:

$$\dot{E}_{1,0} = \dot{E}_{r0} + \dot{E}_{i0} \approx 0$$

But the magnetic field increases about twofold over that of the incident wave, as follows from (9.20). Multiplying the resultant magnetic field by the wave impedance gives the electric field on the metal surface:

$$\dot{E}_{1,0} = \dot{E}_{2,0} = 2\dot{H}_{i0}\zeta_2 = 2\dot{E}_{i0}\zeta_2/\zeta_1 \tag{9.23}$$

Analogy with long transmission lines. It is easy to see the analogy between the above solutions and the corresponding solutions for the case of two long series-connected lines. The wave impedance of a medium, ζ, now plays the role of the wave impedance of the line z_c. Analogy is substantially close if a few boundaries follow in parallel with the first boundary. As with a number of lines connected in series, here, for the calculations of Γ, we must replace the wave impedance of the next medium, ζ_2, by the input impedance ζ_{in} of the entire system that extends beyond the interface.

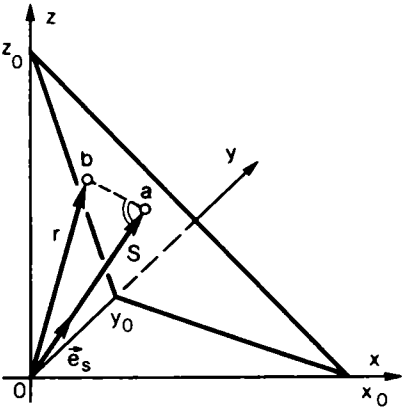

Fig. 9.3

The calculation of ζ_{in} is similar to that of ζ for the system of long lines:

$$\zeta_{in} = \dot{E}_{1,0}/\dot{H}_{1,0} \tag{9.24}$$

where $\dot{E}_{1,0}=\dot{E}_{2,0}$ and $\dot{H}_{1,0}=\dot{H}_{2,0}$ are the electric and magnetic fields at the first boundary.

Wave vector. Let a wave travel along a unit vector $\mathbf{e}_s$ and s be the observation point coordinate counted off from the origin in the direction of $\mathbf{e}_s$ (Fig. 9.3). Based on these assumptions, we can use all the above equations, denoting the coordinate along which the wave propagates by s:

$$\dot{\mathbf{E}} = \dot{\mathbf{E}}_0 e^{-\gamma s} \text{ and } \dot{\mathbf{H}} = \dot{\mathbf{H}}_0 e^{-\gamma s} \tag{9.25}$$

at $\dot{E} = \zeta \dot{H}$.

The propagation constant γ in the above formulas is often expressed in terms of the wave coefficient:

$$k = -j\gamma \tag{9.26}$$

The equations for a traveling electromagnetic wave then assume the form

$$\dot{\mathbf{E}} = \dot{\mathbf{E}}_0 e^{-jks} \text{ and } \dot{\mathbf{H}} = \dot{\mathbf{H}}_0 e^{-jks} \tag{9.27}$$

They differ from (9.25) and preceding equations only in the designation of the exponent.

Referring to (9.10a), it becomes obvious that

$$k = -j\gamma = \omega\sqrt{\underline{\varepsilon}\,\varepsilon_0 \underline{\mu}\,\mu_0} \tag{9.28}$$

For a lossless medium ($\underline{\mu} = \mu, \underline{\varepsilon} = \varepsilon$), the wave coefficient is real:

$$k = 2\pi/\lambda \tag{9.28a}$$

where $\lambda = vT$ or $1/\lambda = f\sqrt{\mu\mu_0\varepsilon\varepsilon_0}$.

In the above expressions, v is the velocity of an electromagnetic wave in a given medium, λ is the wavelength, f is frequency, and $T = 1/f$ is the period.

In describing electromagnetic waves, it is convenient to use the notion of a wave vector. For a plane wave traveling along the $\mathbf{e}_s$ direction, the wave vector is

$$\mathbf{K} = k\mathbf{e}_s \tag{9.29}$$

For any point b, whose position in space is defined by a vector $\mathbf{r}$ drawn from the origin (see Fig. 9.3), we then have

$$\mathbf{re}_s = s \text{ and } \mathbf{rK} = ks \tag{9.30}$$

Proceeding from equalities (9.30), we can replace the exponent in (9.25) and (9.27) by a scalar product

$$-j\mathbf{rK} = -jks = -\gamma s \tag{9.31}$$

A remarkable thing is that the vectors $\dot{\mathbf{E}}$, $\dot{\mathbf{H}}$, and $\mathbf{K}$ form a triad of vectors (Fig. 9.4) related to each other by simple equations:

$$\dot{\mathbf{H}} = \mathbf{K} \times \dot{\mathbf{E}}/\omega\mu\mu_0, \quad \dot{\mathbf{E}} = \dot{\mathbf{H}} \times \mathbf{K}/\omega\varepsilon\varepsilon_0 \tag{9.32}$$

These equations are easy to remember if we keep in mind that $\dot{E} = \zeta \dot{H}$ and the Poynting vector $\underline{\Pi} = \dot{\mathbf{E}} \times \dot{\mathbf{H}}^*$ must be in the direction of wave propagation, i.e. in the direction of $\mathbf{K}$ (see Fig. 8.9).

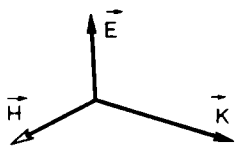

Fig. 9.4

In expression (9.31), the scalar product can be expressed in cartesian coordinates:

$$\mathbf{Kr} = k_x x + k_y y + k_z z \tag{9.33}$$

if $\mathbf{r} = x\mathbf{e}_x + y\mathbf{e}_y + z\mathbf{e}_z$, where k_x, k_y, and k_z are the components of a wave vector in cartesian coordinates. In a lossless case, the relation between the modulus of the wave vector and the moduli of its projections takes a simple form

$$\mathbf{K}^2 = k^2 = k_x^2 + k_y^2 + k_z^2 = \omega^2\varepsilon\varepsilon_0\mu\mu_0 = (2\pi/\lambda)^2 \tag{9.34}$$

The latter part in the equality is nothing else but formula (9.28a) derived above.

Expressing the scalar product $\mathbf{Kr}$ in (9.31) as (9.33), we obtain one more expression for a traveling plane wave:

$$\dot{\mathbf{E}} = \dot{\mathbf{E}}_0 e^{-j(k_x x + k_y y + k_z z)} \tag{9.35}$$

Given the wave vector components, we can easily determine the direction of wave propagation:

$$\mathbf{e}_s = \mathbf{K}/k = (1/k)(k_x\mathbf{e}_x + k_y\mathbf{e}_y + k_z\mathbf{e}_z) \tag{9.36}$$

The velocity with which a free wave advances in space and its phase velocity (see Ch. 8) have the same direction $\mathbf{e}_s$:

$$v = v_{ph,s} = \omega/k \tag{9.37}$$

The wavelength of a free progressive wave is

$$\lambda = vT = v/f = 2\pi/k \tag{9.38}$$

Components of the wave and phase velocities. The wave velocity v can be

represented as a vector

$$\mathbf{v} = v\mathbf{e}_s = v\mathbf{K}/k = v_x\mathbf{e}_x + v_y\mathbf{e}_y + v_z\mathbf{e}_z \tag{9.39}$$

having definite components along the rectangular axes. They are easily found by comparing (9.39) with (9.33) :

$$v_x = v\cos(\mathbf{K}, \mathbf{e}_x), v_y = v\cos(\mathbf{K}, \mathbf{e}_y), \ldots \tag{9.40}$$

Each of the components v_x, ... is smaller than the velocity v in the direction of $\mathbf{K}$. This could be considered obvious and it would be pointless to dwell upon the fact but for the existence of phase velocities in the x, y, and z directions:

$$v_{ph,x} = \Lambda_x f, v_{ph,y} = \Lambda_y f, \ldots \tag{9.41}$$

where Λ_x, Λ_y, ... are the wavelengths in the directions x, y, ... , and $f = 1/T$ is the frequency. We shall prove now that the phase velocities are greater than, or equal to, the velocity of propagation of an electromagnetic wave. This seemingly paradoxical fact relates only to a phase velocity; the propagation velocity of any signal carried by an electromagnetic wave is always lower than, or in the limit equal to, the velocity of the free wave.

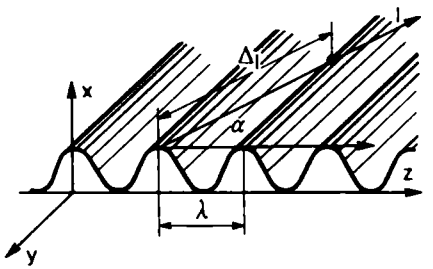

Fig. 9.5

The dependence of the wavelength on the direction of its measurement is specific to all wave processes. Fig. 9.5 illustrates the waves traveling along the z axis. For a z-directed wave, the distance between the crests is minimum and equal to λ. In any other direction, for example, in the l direction, the distance between the crests is larger:

$$\Lambda_l = \lambda/\cos\alpha > \lambda \tag{9.42}$$

where $\alpha = \angle\, \mathbf{e}_z, \mathbf{l}$.

We may refer to Fig. 9.3 to support the point. Let the triangle $x_0y_0z_0$ represent a portion of the plane (normal to the vector $\mathbf{s}$) in which the field vectors are equal to the wave amplitudes. Also, let s be exactly equal to the wavelength. In other words, the adjacent plane proves parallel to the first plane and passes through the origin. The intercepts are then equal to the wavelengths of waves traveling along the corresponding axes:

$$\Lambda_x = x_0, \Lambda_y = y_0, \ldots \tag{9.43a}$$

Since $s = \lambda\mathbf{e}_s$ is normal to the plane, it is obvious that

$$s = \lambda = x_0\cos(\mathbf{s}, \mathbf{e}_x) = y_0\cos(\mathbf{s}, \mathbf{e}_y) = \ldots \tag{9.43b}$$

We have thus arrived at the same formula as (9.42), but expressed in a more

general form. Comparing (9.43) with (9.38) and (9.41) leads us to state that

$$v_{ph,x} = v/\cos(\mathbf{K}, \mathbf{e}_x) \geqslant v$$

$$v_{ph,y} = v/\cos(\mathbf{K}, \mathbf{e}_y) \geqslant v, \ldots \tag{9.44}$$

There is a general relation between v, $v_{ph,x}$ and component v_x in the given direction x. It directly follows from (9.40) and (9.44):

$$v_{ph,x} v_x = v^2 \tag{9.45}$$

It is easy to understand the relation between the phase velocity and wave velocity by considering the known and illustrative physical processes of a wave character. Assume sea waves run to the shore so that the wave crests are slightly nonparallel to the shoreline and form a small angle with it. If we compare a slow motion of the crest of a wave on the sea surface and a high speed with which the foamy crest advances along the shore, we can readily see that the difference in speeds is indeed large. The first corresponds to the wave velocity and the second to the phase velocity along the shore. Fig. 9.6 schematically portrays the sea shore in line with the x axis. The waves whose crests are shown by solid lines move toward the shore at a certain angle α to the shoreline with a speed v.

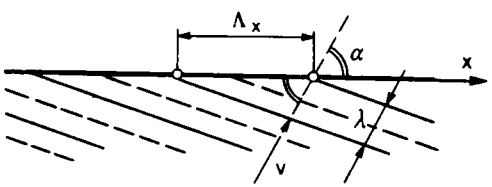

Fig. 9.6

If each of the waves travels a distance λ for a time $T = 1/f$, then any wave will run along the shore a distance

$$\Lambda_x = \lambda/\cos\alpha$$

The smaller the angle the wave crests make with the shoreline, the faster the wave advances along the shore. In the limit, when the waves move normal to the shore (the wave crests are parallel to the shoreline), the phase velocity goes into infinity because one and the same wave phase appears simultaneously along the entire shore. When $v_{ph,x} = \infty$, and thus $\alpha = \pi/2$, the velocity component for the wave moving in the direction of the shore is $v_x = 0$.

Additional notes on the phase velocity and signal propagation velocity are given at the end of Sec. 9.2.

Expression (9.45) applies to the simple case of a single plane wave propagating in a infinite homogeneous medium in the absence of reflected waves. This is a very general expression, and so the phase velocity v_{ph} in a given direction times the signal velocity v_s in the same direction is always equal to the square of the velocity of a free wave:

$$v_{ph} v_s = v^2 \tag{9.45a}$$

Energy transported by a wave. For an incident wave, the Poynting vector according to (9.32) is

$$\underline{\Pi} = \dot{\mathbf{E}} \times \dot{\mathbf{H}}^* = \mathbf{K}E^2/\omega\mu\mu_0 = \mathbf{K}H^2/\omega\varepsilon\ \varepsilon_0 \tag{9.46}$$

The flow of power is in the direction of wave vector **K**, or in the direction of wave propagation.

The density of energy carried by a traveling wave can be found using the known formulas

$$w_e = E^2\varepsilon\varepsilon_0/2,\ w_m = H^2\mu\mu_0/2 \tag{9.47}$$

From the equality $E^2 = \zeta^2 H^2$, i.e. $E^2 = H^2(\mu\mu_0/\varepsilon\varepsilon_0)$, it follows that $w_e = w_m$, and so the total energy density of an electromagnetic wave is

$$w = w_e + w_m = \varepsilon\varepsilon_0 E^2 = \mu\mu_0 H^2 \tag{9.48}$$

If H and E are effective values, then w is the average value of energy density, i.e. the average value in space, along the wave from $E_{\max}$ to $E_{\max}$, or the average value in time, during a period for a stationary point through which the wave passes.

Expression (9.46) for energy density and (9.48) for power flow are related to each other by the equality

$$\underline{\Pi} = w\mathbf{v} \tag{9.49}$$

where **v** is the velocity of a free wave. Really, $v = 1/\sqrt{\mu\mu_0\varepsilon\varepsilon_0}$, and hence

$$wv = H^2\mu\mu_0/\sqrt{\mu\mu_0\varepsilon\varepsilon_0} = E^2\varepsilon\varepsilon_0/\sqrt{\mu\mu_0\varepsilon\varepsilon_0} = EH \tag{9.50}$$

By definition, **K** and **v** are in the same direction.

The above expressions are valid only if the region under consideration is free from a reflected wave or other waves. The principle of superposition applies to field vectors, but is not suitable for energy and power (given as products or quantities squared).

For a plane wave that falls normal to the flat boundary between two media, some simple and interesting relations hold.

Assume the medium of interest is air ($\zeta_1 = \zeta_0, \varepsilon = \mu = 1$) and the wave impinges at a right angle onto a medium displaying any complex parameters. Two waves are then present in the first medium, one traveling forward (the incident wave in the $+ z$ direction), the other backward (the reflected wave in the $- z$ direction). The wave vectors of the incident and the reflected wave are opposite in sign, and are given by (9.18) and (9.20) as

$$\dot{\mathbf{E}}_r = \Gamma\dot{\mathbf{E}}_i,\ \dot{\mathbf{H}}_r = -\Gamma\dot{\mathbf{H}}_i$$

The resultant field vectors according to (9.15) are

$$\dot{E} = \dot{E}_0 e^{-jkz}(1 + \Gamma e^{+j2kz})$$
$$\dot{H} = \dot{E}_0 e^{-jkz}(1 - \Gamma e^{+j2kz})/\zeta_0$$

The Poynting vector is then

$$\underline{\Pi} = \dot{\mathbf{E}} \times \dot{\mathbf{H}}^* = E_0^2[1 - |\Gamma|^2 + j2\ \mathrm{Im}\ (\Gamma e^{+j2kz})]\mathbf{K}/\omega\mu\mu_0 \tag{9.51}$$

The term within the brackets is the difference between conjugates, $\Gamma e^{+j2kz} - \Gamma^* e^{-j2kz}$.

It is interesting that the flow of active power represents the difference between the corresponding components of power flows of the incident and the reflected wave. The imaginary component of power flow, i.e. the reactive power, depends on the coordinate and shows up in the formation of the regions storing predominantly the energy of an electric field and the regions where the energy of a magnetic field predominates. This wave pattern corresponds to that of a standing wave.

Electromagnetic wave and its pressure as the wave impinges on the boundary between two media. In the case of normal incidence of the wave on the flat surface of a well conducting medium, the wave pressure can be found by the method discussed in Sec. 8.1 [see formula (8.37) and Fig. 8.6].

Choose the coordinate system so that the magnetic vector of the incident wave can be equal to $H = H_z$ providing the wave travels along the y axis, as shown in Fig. 9.7.

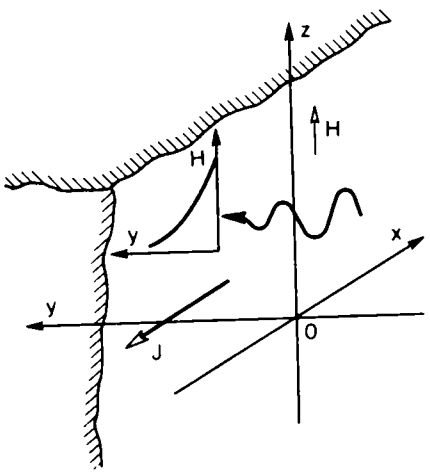

Fig. 9.7

A high-frequency field decays fast inside a metal, producing a current of the density

$$\mathbf{j} = \operatorname{curl} \dot{\mathbf{H}} = \mathbf{e}_x \partial \dot{H}_z / \partial y \tag{9.52}$$

This vector points in the negative direction, $-\mathbf{e}_x$, since H_z decays in the y direction. The metal carrying the current of density $\mathbf{J}$ experiences a force per unit volume

$$\underline{\mathbf{f}}_0 = \mu_0 \mathbf{j} \times \overset{*}{\mathbf{H}} = -\mu_0 \dot{j} \overset{*}{H} \mathbf{e}_y$$

Integrating this force through the entire metal thickness from $y = 0$ to $y = y_0$, i.e. up to the depth of metal where the current density practically goes to zero, we obtain the force related to a unit area of the metal surface, or the pressure

$$\begin{aligned} \underline{\mathbf{F}}_0 &= -\mathbf{e}_y \mu_0 \int_0^{y_0} \dot{j} \overset{*}{H} dy = -\mathbf{e}_y \mu_0 \int_0^{y_0} \overset{*}{H} (\partial \dot{H} / \partial y) dy \\ &= -\mathbf{e}_y \mu_0 \int_{H=H_s}^{H=0} \overset{*}{H} d\dot{H} = \mathbf{e}_y \mu_0 H_s^2 / 2 \end{aligned} \tag{9.53}$$

where H_s is the field strength on the metal surface.

As seen from the formula, the pressure is equal to the energy density of the field on the metal surface. The magnetic field on a well conducting surface is twice that of an incident wave:

$$H_s = 2H_i \tag{9.54}$$

Consequently, the pressure is four times the average magnetic field of an incident wave:

$$F_0 = \mu_0 H_s^2/2 = 4(\mu_0 H_i^2/2) \tag{9.55}$$

or the pressure of an incident wave is a twofold average value of the total energy density (9.48):

$$F_0 = 2(\mu_0 H_i^2/2 + \varepsilon_0 E_i^2/2) \tag{9.56}$$

The same results follow from the general concepts of mechanics in conjunction with the Einstein equation by which the mass density corresponds to the energy density:

$$m_0 = w/c^2$$

The mean mass density of an incident electromagnetic wave is given by

$$m_0 = \mu_0 H_i^2/2c^2 + \varepsilon_0 E_i^2/2c^2 \tag{9.57}$$

A moving mass possesses a momentum equal to the mass times velocity. Hence, the momentum density for an incident wave is

$$\mathbf{g}_0 = m_0 \mathbf{c} \tag{9.58}$$

By Newton's law, a change in the momentum is equal to the impulse of force:

$$d(m\mathbf{v}) = \mathbf{f}dt$$

In our case of normal incidence, where total reflection from the metal surface takes place, the wave velocity becomes opposite in sign. This means that the average momentum density changes by $\Delta g_0 = 2m_0 c$. But the energy flow distributed in column cT bearing on a unit area changes its direction in time T. Therefore, the impulse of force defined as the product of pressure F_0 and time T must be equal to Δg_0 times the length cT:

$$F_0 T = 2m_0 c^2 T \tag{9.59}$$

at

$$F_0 = 2m_0 c^2 \tag{9.60}$$

Considering (9.57) for mass density defined by the Einstein law we arrive at expression (9.56).

Physical processes involved in wave reflection and propagation. Analyzing the behavior of a dielectric in an electric field, we stressed out that a dielectric medium is responsible for the appearance of additional bound charges possesssing an additional field. We also emphasized that a magnetized medium causes the appearance of additional carriers of an electrokinetic (magnetic) moment. In the same manner, a medium certainly has an effect on an electromagnetic wave, namely, on its velocity and direction of propagation. Molecular dipoles vibrating under the action of an electromagnetic wave become the microscopic sources of electromagnetic radiation.

The field of a host of such radiators superposes on the field of an incident wave, forming the resultant field.

9.2. Cavity Resonators and Waveguides

Throughout this book, we have spoken more than once of the guided propagation of waves. The waves traveling between the conductors of cables or overhead lines represent the simplest case of this type of propagation. Electromagnetic waves can also propagate along waveguides (hollow metallic tubes) and fibers of high permittivity, forming what is called a dielectric waveguide. In the first case, waves travel within a tube as they undergo multiple reflections from well conducting walls. In the second case, waves travel by multiple reflections from the boundary between the dielectric and air (the effect of total internal reflection which is well known in optics).

Of much practical significance are cavity resonators. In a way, they resemble acoustic resonators and in the simplest case represent a hollow metallic enclosure for setting up and maintaining spatial standing waves of a strictly definite wavelength. When excited by a field source of a definite frequency, the resonant cavity, like an LC resonant circuit, is capable of storing an appreciable amount of energy. Such cavities have a very high Q, ranging into the tens of thousands.

Different methods are applicable for calculating both waveguides, and cavity resonators. In calculating rectangular resonators and waveguides, one can proceed from the concept of existence of a number of plane waves reflected from well conducting walls. This physical concept of the generation of standing waves due to superposition of a number of traveling waves has certain merits since it is simple and illustrative.

Cavity resonator. Let the metallic walls bounding a rectangular cavity run normal to the coordinate axes, as shown in Fig. 9.8*a*. The distances between parallel walls, i.e. the height, depth, and width of the cavity, are A, B, and C respectively. Assume a wave propagates within this cavity:

$$\dot{\mathbf{E}}_1 = \dot{\mathbf{E}}_0 e^{-j\mathbf{K}_1 \mathbf{r}} \tag{9.61}$$

where $\mathbf{K}_1 = k_x \mathbf{e}_x + k_y \mathbf{e}_y + k_z \mathbf{e}_z$.

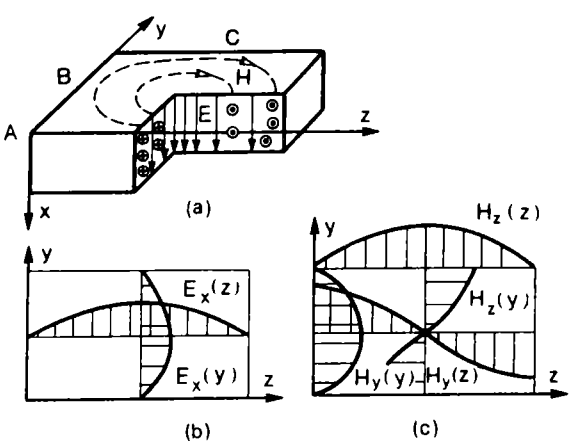

Fig. 9.8

The wave bounces back as it impinges on the wall normal to the x axis. The wave vector of the reflected wave differs from $\mathbf{K}_1$ only by the changed sign of the xth component:

$$\mathbf{K}_2 = -k_x\mathbf{e}_x + k_y\mathbf{e}_y + k_z\mathbf{e}_z \tag{9.62a}$$

As it strikes the wall normal to the y axis, the second wave also bounces back. The wave vector of the new reflected wave differs only by the sign of the yth component:

$$\mathbf{K}_3 = -k_x\mathbf{e}_x - k_y\mathbf{e}_y + k_z\mathbf{e}_z \tag{9.62b}$$

Reasoning in the same way as before, we determine eight possible values of the wave vectors.

As a matter of fact, the condition (9.34) stemming from the basic field equations must be satisfied for each of the vectors. The plane wave specified by any of the wave vectors undergoes a number of reflections and converts to the wave with the same wave vector.

The formulas for the sum of eight different waves, having in the general case different amplitude factors, are rather cumbersome. Indeed, defining each of the exponential factors by use of Euler's formula gives the expression of the form

$$e^{\pm jk_x x} e^{\pm jk_y y} e^{\pm jk_z z}$$
$$= (\cos k_x x \pm j \sin k_x x)(\cos k_y y \pm j \sin k_y y)(\cos k_z z \pm j \sin k_z z) \tag{9.63}$$

The expression consists of eight terms for each set of the chosen signs. Besides, each of the amplitude factors has three terms:

$$\dot{\mathbf{E}} = \dot{E}_x\mathbf{e}_x + \dot{E}_y\mathbf{e}_y + \dot{E}_z\mathbf{e}_z$$

However, all the possible terms can be written in a very compact form since they are of the same type. Thus, the most general form of the expression for the xth component of the electric field becomes

$$\dot{E}_x = \dot{E}_i \left(\begin{matrix}\cos\\ \sin\end{matrix} k_x x\right) \left(\begin{matrix}\cos\\ \sin\end{matrix} k_y y\right) \left(\begin{matrix}\cos\\ \sin\end{matrix} k_z z\right) \tag{9.64}$$

where the subscript i identifies a different combination of factors chosen in the form of cosines and sines.

Equation (9.64) takes a yet simpler form if we use the known boundary conditions for ideally conducting walls: only the normal component of the electric field can be present on the surface of such walls. Hence, E_x must reduce to zero at $y = 0$ as well as at $z = 0$, in which case the expression for E_x contains sines rather than cosines.

We thus can replace (9.64) by the expression of the form

$$\dot{E}_x = \dot{E}_1 \left(\begin{matrix}\cos\\ \sin\end{matrix} k_x x\right) \sin k_y y \cdot \sin k_z z \tag{9.65}$$

In a likewise manner, we obtain

$$\dot{E}_y = \dot{E}_2 \sin k_x x \left(\begin{matrix}\cos\\ \sin\end{matrix} k_y y\right) \sin k_z z$$

$$\dot{E}_z = \dot{E}_3 \sin k_x x \cdot \sin k_y y \left(\begin{matrix}\cos\\ \sin\end{matrix} k_z z\right)$$

Further refinements are easy to perform if we set $E_x = 0$ at $y = B$ and also at

$z = C$ reasoning from the same concept of the field on the surface of a conductor (the walls are parallel to the x axis). The above conditions are met at

$$k_y = m\pi/B \text{ and } k_z = n\pi/C \tag{9.66a}$$

Similar conditions on the yth and zth components lead us to the equality

$$k_x = l\pi/A \tag{9.66b}$$

In the three latter equalities, the factors l, m, and n are integers called characteristic numbers. A set of these numbers, such as $l = 0, m = 1$, and $n = 1$ or $l = 1, m = 1$ and $n = 0$, etc. characterizes oscillation modes.

Equalities (9.66) in combination with (9.34) enable us to arrive at the equation relating the frequency f or the wavelength of a free progressive wave, $\lambda = c/f$, to the dimensions of a resonator:

$$(k/\pi)^2 = (2/\lambda)^2 = (l/A)^2 + (m/B)^2 + (n/C)^2 \tag{9.67a}$$

or

$$(2\pi/\lambda)^2 = k_x^2 + k_y^2 + k_z^2 \tag{9.67b}$$

This means that a certain maximum wavelength (or minimum frequency f) is specific to a resonator of given dimensions A, B, and C. At this wavelength, resonance oscillations can be excited inside the cavity; in other words, an alternating electromagnetic field can be induced in the form of spatial standing waves. The oscillations at which the electric vector points in the direction of the shortest cavity side correspond to a maximum wavelength. Thus by (9.67) we have

$$\lambda/2 = 1/\sqrt{(l/A)^2 + (m/B)^2 + (n/C)^2} \tag{9.68}$$

If $A < B \leqslant C$, then it is evident that $\lambda = \lambda_{max}$ at $l = 0$.
The other two integers must be of the lowest possible value, $m = n = 1$. The electric vector (9.65) is then given by

$$\dot{E} = \dot{E}_x = \dot{E}_1 \sin(\pi y/B) \sin(\pi z/C) \tag{9.69}$$

The other components $\dot{E}_y$ and $\dot{E}_z$ are equal to zero (see Fig. 9.8)[1]. Under these conditions

$$\lambda = \lambda_{max} = 2BC/\sqrt{B^2 + C^2} \tag{9.70}$$

At $A < B = C$,

$$\lambda_{max} = B\sqrt{2}$$

Formulas (9.65) are subject to further refinement since each of the rows still contains a factor in which we must choose a cosine or sine. It can be shown in a variety of ways that we must fix upon cosines and, hence, replace (9.66) and write:

$$\begin{aligned} \dot{E}_x &= \dot{E}_1 \cos k_x x \cdot \sin k_y y \cdot \sin k_z z \\ \dot{E}_y &= \dot{E}_2 \sin k_x x \cdot \cos k_y y \cdot \sin k_z z \\ \dot{E}_z &= \dot{E}_3 \sin k_x x \cdot \sin k_y y \cdot \cos k_z z \end{aligned} \tag{9.71}$$

---

[1] If, in addition, we set m or n equal to zero, the xth component vanishes too.

The simplest way of obtaining this result is to calculate $\operatorname{div}\dot{\mathbf{E}}$ which must be equal to zero in the absence of a space charge. Using (9.71), we find that

$$\begin{aligned}\operatorname{div}\dot{\mathbf{E}} &= \partial\dot{E}_x/\partial x + \partial\dot{E}_y/\partial y + \partial\dot{E}_z/\partial z \\ &= -(\dot{E}_1 k_x + \dot{E}_2 k_y + \dot{E}_3 k_z)\sin k_x x \cdot \sin k_y y \cdot \sin k_z z\end{aligned}$$

One of the sines identically reduces to zero when any of the integers l, m, or n is equal to zero. Here, as is required, $\operatorname{div}\dot{\mathbf{E}} = 0$, and so no restrictions are imposed on the relation between field vectors. If any of the integers does not go to zero, then the relation between the amplitude factors enclosed in the parentheses must be such that the resultant value is equal to zero. A different choice of cos and sin in (9.65) leads to the set of equations that has a unique solution, $E_1 = E_2 = E_3 = 0$.

The magnetic field inside a cavity can always be found from Maxwell's second equation on condition, of course, that $\dot{\mathbf{E}}$ is known:

$$\dot{\mathbf{H}} = (j/\omega\mu_0)\operatorname{curl}\dot{\mathbf{E}} \tag{9.72}$$

Consider the field pattern in the simplest case when $l = 0, m = 1, n = 1$ (see Fig. 9.8*b* and *c*). According to (9.69) and (9.72), we have

$$\begin{aligned}\dot{E} = \dot{E}_x &= \dot{E}_1 \sin(\pi y/B)\sin(\pi z/C) \\ \dot{H}_x &= 0 \\ \dot{H}_y &= (j/\mu_0\omega)(\pi/C)\dot{E}_1 \sin(\pi y/B)\cos(\pi z/C) \\ \dot{H}_z &= (-j/\mu_0\omega)(\pi/B)\dot{E}_1 \cos(\pi y/B)\sin(\pi z/C)\end{aligned} \tag{9.73}$$

To excite oscillations inside a resonant cavity, the external source of oscillations must properly be coupled to the chamber. For this, a hole is made in the inner wall of the resonator and a coaxial cable (or a waveguide) is inserted through this hole into the enclosure.

With an electrical method of excitation, the cable conductor is inserted to a small depth into the chamber (Fig. 9.9*a*). The electric field pattern of the conductor inside the chamber must, at least partially, resemble the configuration of the field to be induced.

With a magnetic method of excitation, the conductor is bent in the form of a small loop and its end is made fast to the chamber wall (Fig. 9.9*b*). The configuration of the magnetic field of the loop must partially resemble the configuration of the field to be induced.

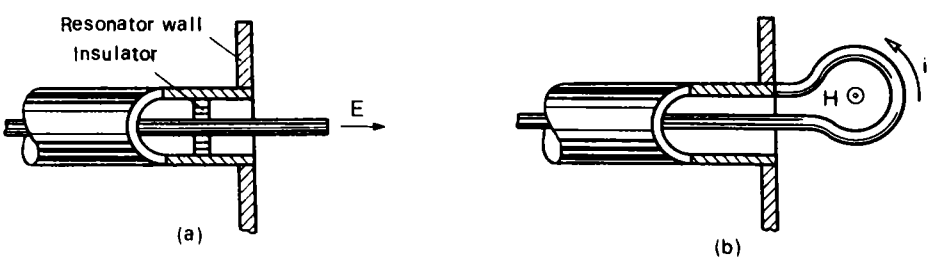

Fig. 9.9

In electric excitation of the field ($l = 0, m = n = 1$), the cable must pass through the hole in the wall normal to the x axis and the post extending into the cavity must be parallel to the x axis (see also Figs. 9.14, 9.15, and the pertinent text). In magnetic excitation of the same field, the cable must be connected to the wall parallel to the x axis.

Losses in and the quality factor of resonators. As mentioned earlier, the Q factor of resonators is very high and runs into the tens of thousands for a wavelength of 10 cm.

Losses can roughly be estimated from the known values of the magnetic field on the wall surface, using formulas (8.61c):

$$P_0 = \mathrm{Re}(H^2\zeta) = H^2/\sigma x_0 = H^2\sqrt{\omega\mu\mu_0/2\sigma} \tag{9.74}$$

where H is the component tangent to the surface (effective value), and $x_0 = \sqrt{2/\omega\mu_0\mu\sigma}$ is the conditional depth of penetration.

Formula (9.74) expresses power referred to a unit area of the surface. Taking the integral of P_0 over all the six walls, we can calculate the dissipated power.

To determine the Q, we must also know the energy stored in the resonator. It can readily be found by integrating over the volume the value of a maximum energy density of the electric field (if E is an effective value):

$$W_{e\,\max} = \int E^2\varepsilon_0 dV$$

The maximum magnetic field energy is of the same magnitude because twice in one cycle all the electric field energy converts to the magnetic field energy. What attests to this statement is the general concept of oscillations in a resonator and also the shift in phase by $\pi/2$ between the electric and magnetic field vectors, as follows from the comparison of formulas (9.73). The ratio between the stored energy and dissipated power expresses the quality factor:

$$Q = \omega W_{e\max}/P \tag{9.75}$$

Application of cavity resonators. We have considered a simple resonator and a simple configuration of the field excited in its cavity. Exciting other modes calls for proper dimensioning of a resonator and more complex coupling means for jointing the external source to the chamber.

Cavity resonators largely operate in the microwave region at wavelengths from about 10 cm to a few millimeters. At greater wavelengths, the dimensions of cavities become too large, and so it is more practicable to use portions of lines as resonators tuned to a desired wavelength.

Rooms for hf apparatus and hf generators are often surrounded by a metallic shield and thus are capable of storing standing waves of a certain frequency. The fields of high strength that are sometimes set up in such rooms can be harmful to the health of attendants. To preclude the deleterious effect of high fields, generators must have an additional shield. In some cases the metallic walls must be provided with an absorption coat.

At wavelengths below a millimeter, cavity resonators and waveguides prove impractical because very small parts require highly precise manufacturing techniques. Besides, tuning and handling of such resonators involves difficulties. Recent ad-

vances in moletronics have brought into wide use millimetric and submillimetric wave bands. The branch of engineering that deals with so short waves becomes rather closely linked with optics. In use are now many devices which inherently differ in design even from microwave devices. These are first of all open resonators and open waveguides. What distinguishes these devices is that their dimensions noticeably exceed electromagnetic wavelengths, just as the dimensions of mirrors and lenses in optics greatly exceed optical wavelengths. In the simplest case, an open resonator is just a combination of two metallic mirrors disposed one opposite the other. These resonators are adapted for use in quantum generators.

Rectangular waveguide. Consider a rectangular waveguide (Fig. 9.10) having dimensions A and B along the x and y axes and infinitely extending in the direction of the z axis. The configuration of the field induced in such a waveguide differs from the field configuration in the rectangular resonator of Fig. 9.8*a* only in that the waves do not undergo reflections from the wall normal to the z axis. In the set of wave vectors (9.63), therefore, the component k_z is of one sign only.

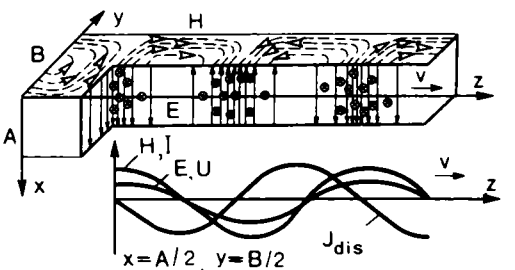

Fig. 9.10

For a + z-directed wave, the exponent of e just becomes $-jk_z z$ (rather than $+jk_z z$). The possible solutions must then differ from (9.71) since in these formulas we should replace $\sin k_z z$ and $\cos k_z z$ by $e^{-jk_z z}$:

$$\begin{aligned} \dot{E}_x &= \dot{E}_1 \cos k_x x \cdot \sin k_y y \cdot e^{-jk_z z} \\ \dot{E}_y &= \dot{E}_2 \sin k_x x \cdot \cos k_y y \cdot e^{-jk_z z} \\ \dot{E}_z &= \dot{E}_3 \sin k_x x \cdot \sin k_y y \cdot e^{-jk_z z} \end{aligned} \tag{9.76}$$

The form of (9.67) turns out to be the same as before

$$k_x^2 + k_y^2 + k_z^2 = (2\pi/\lambda)^2$$

Also,

$$k_x = l\pi/A, k_y = m\pi/B \tag{9.77}$$

In order that the electromagnetic waves may propagate in the z direction without attenuation, k_z should be real; in other words, the following inequality should be satisfied:

$$k_z^2 = (2\pi/\lambda)^2 - (l\pi/A)^2 - (m\pi/B)^2 > 0 \tag{9.78}$$

This condition shows that the waveguide can transmit waves whose wavelength in free space is below a certain critical value:

$$\lambda < \lambda_{cr} = 2/\sqrt{(l/A)^2 + (m/B)^2} \tag{9.79}$$

A critical wavelength depends on the mode to be set up, each mode being defined by the number l and m. If $A < B$, then the largest critical wavelength, $\lambda_{cr\max} = 2B$, corresponds to numbers $l = 0$ and $m = 1$; both l and m cannot reduce to zero at the same time, as is apparent from (9.76) and (9.77).

At $l = 0$ and $m = 1$, we have $k_x = 0$ and $k_y = \pi/B$. Hence, according to (9.76),

$$\dot{E} = \dot{E}_x = \dot{E}_1 \sin(\pi y/B) e^{-jk_z z} \tag{9.80}$$

The magnetic field of a wave can easily be found by Maxwell's equation

$$\dot{\mathbf{H}} = (j/\omega\mu_0) \operatorname{curl} \dot{\mathbf{E}}$$

At $l = 0$ and $m = 1$, we obtain

$$\begin{aligned} \dot{H}_x = 0, \dot{H}_y &= -jk_z \dot{E}_1 \sin k_y y \cdot p \\ \dot{H}_z &= -k_y \dot{E}_1 \cos k_y y \cdot p \end{aligned} \tag{9.81}$$

where $p = je^{-jk_z z}/\omega\mu_0$.

Wavelength Λ in a waveguide and phase velocity. Knowing the zth component of the wave vector, we find Λ of the wave propagating in the waveguide:

$$\Lambda = 2\pi/k_z \tag{9.82}$$

Comparing (9.78) with (9.82) we can conclude that this wavelength is always larger than λ of a free wave. This is evident if we reduce (9.82) to the form

$$\Lambda = \lambda/\sqrt{1 - (\lambda/\lambda_{cr})^2} \tag{9.83}$$

The phase velocty v_{ph} of a wave traveling along the waveguide is thus always higher than the wave velocity in free space:

$$v_{ph} = \Lambda f > v = \lambda f \tag{9.84}$$

The relation between λ and Λ is shown schematically in Fig. 9.11 which illustrates the wave that propagates along a waveguide as it undergoes multiple reflections from lower and upper walls. The figure also gives one more quantity, λ_z, which specifies the distance the crest travels during a period T along the z direction, i.e. along the wave vector of the incident wave. This path is smaller than Λ and hence more so than λ. The wavelength Λ grows with α, but λ_z decreases.

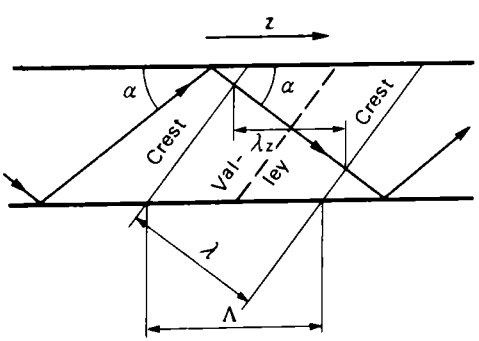

Fig. 9.11

The relation between these three wavelengths is simple:

$$\Lambda\lambda_z = \lambda^2 \tag{9.85}$$

For the case shown in Fig. 9.11, this is obvious since $\lambda_z = \lambda\cos\alpha$ and $\Lambda\cos\alpha = \lambda$. If we divide both sides of equality (9.85) by T^2, we shall find that

$$v_{ph}v_z = v^2 \tag{9.86}$$

where v is the velocity of a wave in free space, equal to the velocity of light, and v_{ph} is the phase velocity of a wave in the z direction.

What meaning does velocity v_z have? To give the answer to this question, let us first solve the following problem. If we assume that the wave has traversed a cross section *1* of the waveguide at some instant, what time does the wave take to cover a path $z_{12} = s$ to traverse a cross section *2*? The wave will cover this distance in a time

$$\tau = (s/\cos\alpha),\ v = s/v\cos\alpha$$

Indeed, the wave travels to the second cross section not in a straight path but by a zigzag way, now being reflected from one wall, now from the other, etc. Since the wave falls at an angle α, the distance run by the wave is equal to $s/\cos\alpha$. The velocity $v\cos\alpha = v_z$ is the velocity of propagation of a signal in the z direction. Obviously, this velocity cannot be higher than the velocity v of a free wave; v_z is equal to v if the z axis is in the direction of propagation of a free wave.

We can get a clearer idea of the velocity v_z as a signal propagation velocity by visualizing a short optical signal sent by a projector in a direction that makes an angle α with the z axis. This signal will travel in the z direction by multiple reflections from the walls. The relation (9.86) is valid for all cases involving three velocities as well as for the case of a rectangular waveguide (see also the text at the end of this section).

The sketch of Fig. 9.12 is another attempt to show more vividly the configuration of an electric field at a certain instant t. All the field patterns shift in time along the z axis with a velocity $v_{ph} = \Lambda/T = \Lambda f$. The field map (grid) becomes such as shown in the figure as a result of multiple reflections at $l = 0$ and $m = 1$, or, which is the same, at $k_x = 0$, $k_y = \pi/B$. The level of the grid in Fig. 9.12 represents the instantaneous value of vector $E = E_x$.

The field strength at the given instant is negative where the grid is below the zero level. The grid has the zero level near the waveguide sides ($y = 0$ and $y = B$) where $E_x = 0$ according to the condition. The distance between the two neighboring tops of the grid (along the z axis) represents the waveguide wavelength Λ.

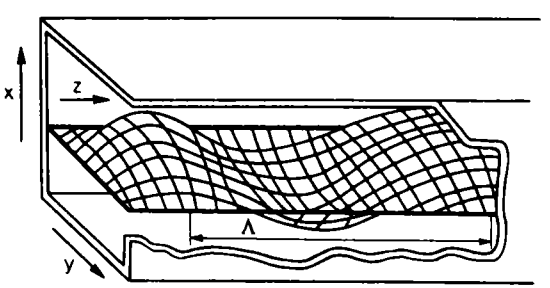

Fig. 9.12

TE and TM waves. Expressions (9.76) and (9.78) deduced above describe in the most general way a variety of waves propagated in rectangular waveguides. They are mainly divided into two classes. To the first belong the waves having no axial component, i.e. the zth component of the electric vector ($\dot{E}_z = 0$). Such waves (fields) are known as transverse electric (TE) waves (modes), or magnetic (H) waves (modes), the two terms being synonymous. The second class includes waves which have no axial magnetic component. These are transverse magnetic (TM) waves, or electric (E) waves.

Waves are further classified as certain modes according to the values of integers; the wave is said to be a TE_{lm} mode where $E_3 = 0$ and the integers are l and m, or a TM_{lm} mode if $H_z = 0$.

The wave depicted in Figs. 9.10 and 9.12 is the wave of the TE_{01} mode. This wave has an electric vector defined by (9.80) and a magnetic vector with the yth and zth components given by (9.81).

For TE_{02} modes, the field configuration features not one but two half-waves of the sinusoid E_x (and the corresponding magnetic field) present along the y axis, as shown in Fig. 9.13.

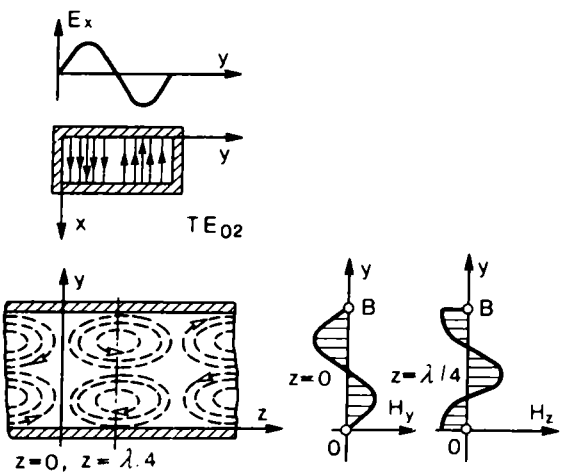

Fig. 9.13

Comparing the fields of a waveguide and a resonator (with a standing wave stored) reveals that in the resonator the orthogonal vectors $\dot{\mathbf{E}}$ and $\dot{\mathbf{H}}$ are $\pi/2$ out of phase and the real component of the Poynting vector is zero. This is not the case for the waveguide with a traveling wave, where the transfer of energy takes place in the same manner as in a long transmission line.

The point worthy of note is that waveguides are analogous in many respects to long lines; namely, the expressions for waves traveling in a waveguide include the factor $e^{-jk_z z}$. A large number of the rules in the theory of long lines apply to waveguides. Of significance is the fact that the notion of an input and a characteristic impedance and also the notion of a reflection coefficient are also suitable for the description of waveguides.

The characteristic impedance depends on the mode under consideration. For the TE_{01} mode, this impedance takes the form

$$Z_c = \dot{E}_x/\dot{H}_y \tag{9.87}$$

By use of (9.80) and (9.81), we find that

$$Z_c = \omega\mu_0/k_z = \sqrt{\mu_0/\varepsilon_0/\sqrt{[(1 - \lambda/\lambda_{cr})^2]}} \tag{9.88}$$

As λ approaches λ_{cr}, Z_c grows. Waveguides find use in the measurements of electrical and magnetic properties of a medium over a wide frequency range including a millimetric wave band.

Waves are reflected from the waveguide end, whether it is open or closed. A part of energy is radiated into space through the open end. At its open end, the waveguide is furnished with a horn, which very closely resembles an acoustic horn, to decrease the reflection from the open end and increase the amount of radiant energy.

Excitation of various modes. The mode of a wave sustained in a waveguide largely depends on the type of coupling of the waveguide to the generator. Fig. 9.14 shows a post inserted in the waveguide for exciting the TE_{01} wave mode.

For excitation of the TE_{02} mode, the waveguide should be complete with two posts; their fields must differ in sign and must be displaced across the waveguide (along the y axis) so that each post is in the center of circles of magnetic lines illustrated in Fig. 9.13.

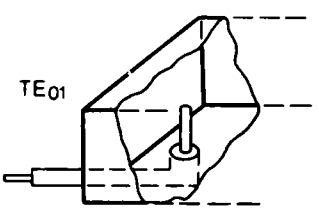

Fig. 9.14

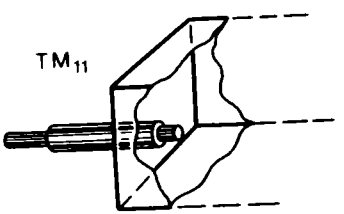

Fig. 9.15

The type of connection of the cable to the waveguide, as shown in Fig. 9.15, serves to induce the TM_{11} wave. In this wave, the electric field has an axial component, as is apparent from the method of excitation. But the magnetic field has only a transverse component (the TM, or E wave). The field configuration for the TM_{11} mode is shown in Fig. 9.16. The connector for jointing the source to the waveguide can also be in the form of a loop whose magnetic field must (partially) resemble in configuration the magnetic field of the wave being excited.

For various wave modes, the waveguide cutoff frequency (critical frequency, or guide wavelength) is different in magnitude. Waveguides excited in a single mode at a given frequency of the generator are most practical.

Summary. In modern practice, the fields of uses of electromagnetic waves are most diverse. Comparatively long waves find use in radio communication (wavelengths from thousands of meters to several meters) and in television (about a meter). In radio engineering, radio links also enjoy use. In such links, a coaxial cable or a

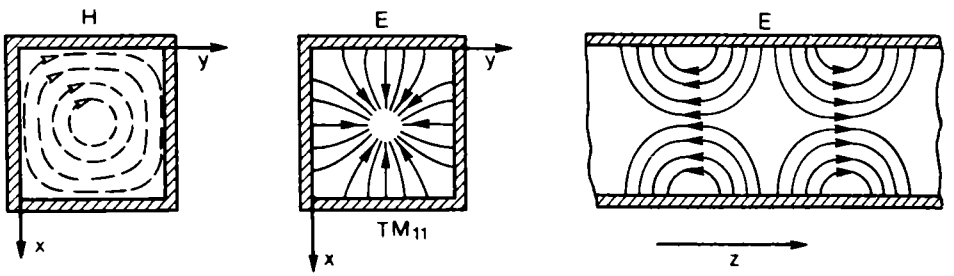

Fig. 9.16

waveguide delivers electromagnetic waves commonly of a wavelength from 3 m to 3 cm to concave mirrors or horn antennae from which the waves propagate (as a projector ray) along the straight line to a mast with receiving mirrors or horns. After amplification, the beam of electromagnetic waves is sent to the next mast.

The earth's man-made satellites can serve as relay stations.

Of interest are some more notes on the behavior of waves of different wavelengths: (1) long waves remain tied to the earth's surface that serves as an open waveguide for them; (2) short waves reach remote points as they undergo reflection from conducting layers of the ionosphere: (3) still shorter waves freely pass through the ionosphere, and so their reception is possible only in the field of "direct vision". It is these waves that are transmitted by chain relaying, including retransmission from satellites.

In conclusion of this digression on radio engineering, let us point out that an increase in wave frequencies is of advantage in radio communication since more and more information (by telephone, television, etc.) can be conveyed over a small range of frequencies. Therefore, laser-beam communication offers exceptionally wide opportunities.

Decimetric and centimetric wave bands find use in radiolocation.

The waves covering the millimetric and submillimetric wave bands enjoy important uses in physicochemical investigations. Still shorter waves approach the optical range.

Phase velocity and a collecting lens. It is sometimes said that the velocity of a free wave is indeed the wave velocity (by the Einstein law, it can never exceed the velocity of light in vacuum), whereas the phase velocity is the notion that owes its origin to a purely formal definition and has no real meaning. The latter argument is not wholly correct because the formal definition of phase velocity v_{ph} stems from the concept of real phenomena and can be applied in the calculation and description of real processes. To illustrate the point, let us compare an ordinary optical collecting lens (Fig. 9.17*a*) with a collecting lens (Fig. 9.17*b*) consisting of a number of parallel or coaxial conducting plates. In the latter, the phase velocity v_{ph} is greater than the light velocity c between two conducting surfaces (as in a waveguide with parallel walls), so that the thickness of the lens grows with distance from the center. On the contrary, the thickness of the ordinary optical collecting lens must decrease with distance from the center since the wave velocity in glass is lower than that in the air.

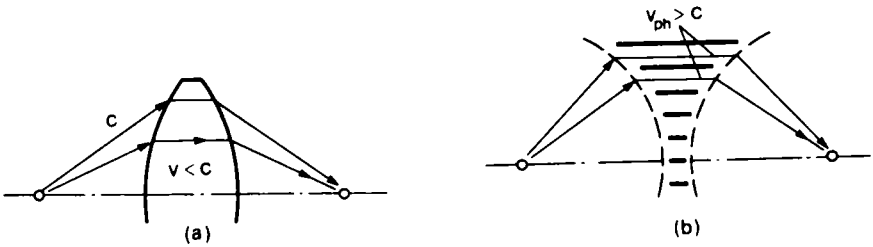

Fig. 9.17

Other notes on various velocities. If a medium does not display dispersion, and so its ε, μ, and hence v of a free wave propagating in this medium are independent of frequency, there is no need to introduce some other velocities apart from v, v_{ph}, and signal velocity v_s. For a dispersive medium whose properties are frequency dependent, i.e. $\varepsilon(j\omega)$ and $\mu(j\omega)$, it is necessary to use additional velocities such as energy flow velocity u and the rate of energy transfer, u_1. In a nondispersive medium, these two types of velocity coincide with signal velocity: $u = u_1 = v_s$.

9.3. Waves in a Nonlinear Medium

The analysis of propagation of waves in nonlinear media presents a complex and important problem in electrodynamics. We shall dwell here only on the qualitative aspect of origin of shock waves in a nonlinear ferromagnetic dielectric.

The wave velocity is inversely proportional to the square root of permeability. The permeability proper decreases with rising field strength due to saturation. Hence, the portions of an incident wave that have a higher amplitude of the magnetic field travel faster.

Consider a wave in a coaxial cable whose insulation consists of ferrite. Assume that at the sending end of the line the wave has the shape as shown in Fig. 9.18*a*. What shape will the wave acquire in time Δt noting that the velocity of the wave grows with an increase in its magnetic field?

The answer is given in Fig. 9.18*b*. The wave portions with the highest H have outrun the rest of the portions with a lower field strength so that the wave is seen to

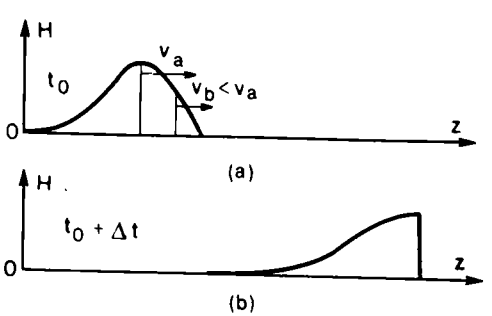

Fig. 9.18

have a very steep front indicative of an almost instantaneous increase in the field of the wave to its maximum value.

This process is reminiscent of turnover of a sea wave as it approaches the shore. The lower layers of water put in motion by the running wave lose speed because of their proximity to the bottom and the upper layers of the wave outstrip the wave base, forming a ridge hanging over the wave. A similar process is encountered in aerodynamics during the formation of shock waves.

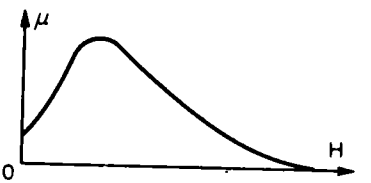

Fig. 9.19

Under certain conditions (at definite H amplitudes for a given magnetic characteristic), the permittivity may grow with rising field; this is evident from the curve of permittivity in weak fields (Fig. 9.19). What will happen in this case? Obviously, the forward portions of the wave will break away and go ahead (until they reach a maximum); the energy they carry will spread along the line, and the wave will have its forward part so smeared that it can hardly be discernible. Such a process also aids in the buildup of a steep front. A most interesting feature in this process is that the tail must catch up: at weak H fields, the velocity is higher due to a lower permeability. It can be expected that in the end the wave of Fig. 9.20*a* will acquire the shape as illustrated in Fig. 9.20*b*. Fig. 9.21 shows a shaper of rectangular pulses, suggested in the USA. It consists of a number of nonlinear two-ports the theory of which was developed by Melville. From the theory of distributed circuits it is known that a line can be represented by a circuit of two-ports. As shown in Sec. 9.1, the propagation and reflection of plane waves follow the laws analogous to the ones describing transmission lines. It is thus clear that the pulse shaper of Fig. 9.21 is nothing else than a circuit model of a long line with nonlinear parameters. The voltage pulse at the

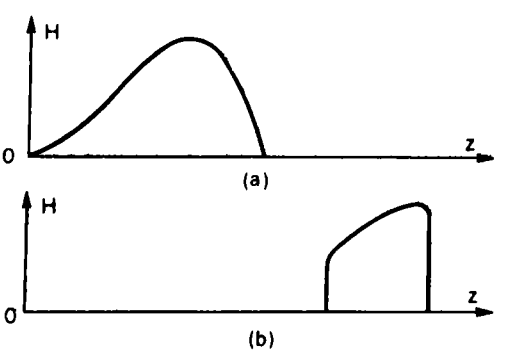

Fig. 9.20

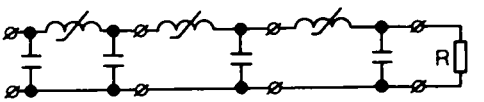

Fig. 9.21

output of the shaper deforms in shape as compared to the input pulse in the same way as the wave does when it propagates in a medium with nonlinear parameters.

It is also instructive to point to the similarity and difference between the circuits of Figs. 8.16 and 9.21. The first is a model of wave propagation in a nonlinear conducting medium, and the second in a nonlinear dielectric medium. The first circuit thus has resistors inserted into its transverse branches, and the second capacitors. Nonlinear effects can result from electric as well as magnetic nonlinearity. Semiconductors are an example.

Chapter 10 Electromagnetic Wave Radiation

10.1. Introductory Remarks

In the previous chapters we often focused on the behavior of waves of an electromagnetic field. Maxwell's conception of why a field propagates in free space with light velocity c was dealt with in Ch. 1. The description of wave propagation and reflection was given in Ch. 9. But up to this point we did not touch upon the topic of how waves originate, i.e. how oscillating currents and charges bring about the emission of energy in the form of electromagnetic waves. The thing is that this topic involves complicated mathematical analysis. Now that we have acquired experience in handling the field equations and performing mathematical operations, we can look into the subject matter and understand more comprehensibly the essence of mathematical transformations.

It is easy to get a qualitative conception of electromagnetic radiation by considering any mechanical vibrations, for example, ripples in the water that spread out in circles from the place where a stone was thrown. Radially propagating waves whose amplitude dies out as they run away evidence that the stone has given up part of its energy to the waves, and this energy will never return to the place where the stone fell.

The energy carried by the ripples spreads out over the surface of water[1].

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[1] In this connection, it might be to the point to recall Shakespeare's verse. Says Joan of Arc:

> . . . like a circle in the water,
> Which never ceaseth to enlarge itself
> Till by broad spreading it disperse to naught.
> (*Henry VI*).

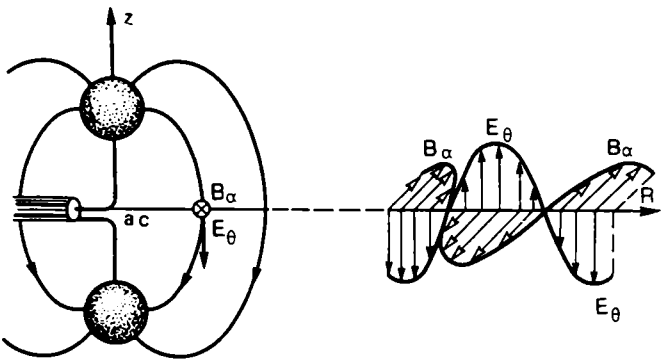

Fig. 10.1

It might be useful to consider one more example. If we immerse a stick into water and then raise it vertically with a constant speed, we can see that the level of water slightly rises round the stick. If we now lift up and let down the stick periodically at a definite rate, we can watch ripples radially running away over the surface of water, which are quite discernible at a large distance from the stick. This effect bears analogy to the radiation of electromagnetic waves.

The electric field of a dipole shown in Fig. 10.1 decays fast with distance R. Indeed, the field potential [see Eq. (1.24)] can be written as

$$\varphi = q\mathbf{l}\mathbf{R}/4\pi\varepsilon_0 R^3 = q\mathbf{l}\mathbf{e}_R/4\pi\varepsilon_0 R^2 \tag{10.1}$$

and the field components [see Eq. (1.25)] are given by

$$E_R = ql\cos\theta/2\pi\varepsilon_0 R^3$$
$$E_\theta = ql\sin\theta/4\pi\varepsilon_0 R^3$$

The field strength decays inversely with the cube of the distance.

If the dipole moment $q\mathbf{l}$ varies fast, for example, the dipole draws power from an ac generator, then the field due to this varying dipole spreads out noticeably farther, decays comparatively slower, and displays in effect the features of an electromagnetic wave. For example, let the dipole moment $ql = q_m l\cos\omega t$ vary in a simple sinusoidal pattern. In the plane normal to the dipole that passes midway between the spheres, we have at $l = l_z$ and at a distance R:

$$B = B_\alpha = (\mu_0/4\pi)(\omega^2/Rc)q_m l\cos\omega\,(t - R/c)$$
$$E = E_\theta = cB$$

where c is the velocity of light (the velocity of an electromagnetic wave in vacuum).

The field strength now decreases inversely as the first power of the distance.

10.2. Delayed Potentials

In the calculation of radiated electromagnetic waves, it is generally more convenient to deal with a scalar potential φ and vector potential $\mathbf{A}$ rather than with the field strength. But the relation of these potentials to charges and currents is known only for static and quasistatic fields:

$$\varphi = (1/4\pi\varepsilon_0)\int \rho\, dV/R$$
$$\mathbf{A} = (\mu_0/4\pi)\int \mathbf{J}\, dV/R \tag{10.2}$$

[see Eqs. (1.21a) and (2.62a)].

It is therefore necessary to find, where possible, similar relations for charges and currents whose densities $\rho(t)$ and $\mathbf{J}(t)$ change fast.

Refer to the field equations deduced for a vacuum:

$$\text{curl } \mathbf{B} = \mu_0\mathbf{J} + \mu_0\varepsilon_0\partial\mathbf{E}/\partial t$$
$$\text{curl } \mathbf{E} = -\partial\mathbf{B}/\partial t \tag{10.3}$$
$$\text{div } \mathbf{E} = \rho/\varepsilon_0,\ \text{div } \mathbf{B} = 0$$

Expressing the main field vectors in the above equations in terms of potentials

$$\mathbf{E} = -\text{grad}\,\varphi - \partial\mathbf{A}/\partial t,\quad \mathbf{B} = \text{curl } \mathbf{A} \tag{10.4}$$

and substituting them in the third and the first equation of the set of equations (10.3), we obtain

$$\text{div } \mathbf{E} = -\nabla^2\varphi - \text{div}\,\partial\mathbf{A}/\partial t = \rho/\varepsilon_0 \tag{10.5}$$

$$\text{curl } \mathbf{B} = \text{curl curl } \mathbf{A} = -\nabla^2\mathbf{A} + \text{grad div } \mathbf{A}$$
$$= \mu_0\mathbf{J} - (1/c^2)(\partial/\partial t)\text{grad}\varphi - (1/c^2)(\partial^2\mathbf{A}/\partial t^2) \tag{10.6}$$

We have used here the identity transform curl curl $\mathbf{A} = -\nabla^2\mathbf{A}$ + grad div $\mathbf{A}$ and written $1/c^2$ instead of $\varepsilon_0\mu_0$.

In the latter equality, it is expedient to transpose all the terms containing the potential to the left-hand side. The expression then takes the form

$$-\nabla^2\mathbf{A} + (1/c^2)(\partial^2\mathbf{A}/\partial t^2) + \text{grad}\,[\text{div } \mathbf{A} + (1/c^2)(\partial\varphi/\partial t)] = \mu_0\mathbf{J} \tag{10.7}$$

The sequence of differentiating with respect to space coordinates and to time can be changed, and therefore

$$(\partial/\partial t)\ \text{grad}\ \varphi \equiv \text{grad}\ \partial\varphi/\partial t$$

Expressions (10.5) and (10.7) contain both the scalar and the vector potential. This appears to lead to much inconvenience, for the scalar potential in (10.5) seems to be related not only to ρ but also to $\mathbf{A}$. Also, the vector potential in (10.7) seems to be related both to $\mathbf{J}$ and to the scalar potential. But these seeming relations are easy to eliminate. It will be recalled that in defining the vector potential (see Sec. 2.5), we pointed to the possibility of specifying arbitrarily the values of div $\mathbf{A}$. Thus, we can always add such an arbitrary function of coordinates and time, grad λ, to the expression of vector potential $\mathbf{A}$ that the divergence of $\mathbf{A}$ takes a desirable form. The

added function cannot affect curl **A** since the curl of any gradient is identically equal to zero, curl grad $\lambda = 0$.

In order for (10.5) and (10.7) to be homogeneous, which greatly simplifies their solution, it is enough to assume that in vacuum

$$\operatorname{div} \mathbf{A} = -(1/c^2)(\partial\varphi/\partial t) \tag{10.8}$$

and in a dielectric with ε and μ

$$\operatorname{div} \mathbf{A} = -(\varepsilon\mu/c^2)(\partial\varphi/\partial t) \tag{10.8a}$$

With these assumptions made, the vector potential in (10.5) vanishes and, besides, the sum within the brackets in (10.7) reduces to zero.

The possibility of simplifying Eq. (10.7) by an appropriate choice of the expression for div **A** stimulates us to supplement the set of Maxwell's equations by Eq. (10.8) which is known as the gauge condition or Lorentz condition[1].

Eqs. (10.5) and (10.7) for potentials then take on a symmetric form

$$\nabla^2\varphi - (1/c^2)(\partial^2\varphi/\partial t^2) = -\rho/\varepsilon_0 \tag{10.9}$$

$$\nabla^2\mathbf{A} - (1/c^2)(\partial^2\mathbf{A}/\partial t^2) = -\mu_0\mathbf{J} \tag{10.10}$$

They differ from the known equations for stationary and slowly varying fields only by the second terms on the left-hand side. For a stationary field, these terms reduce to zero. They are also negligible for a slowly varying field (the second derivative divided by the light velocity squared). Eq. (10.9) was first deduced by Riemann in 1858, who applied it to describe induction phenomena of electric currents (Maxwell's treatise, sec. 862).

The equations given above are wave equations. The simplest one-dimensional wave equation received much scrutiny in Chs. 8 and 9:

$$\partial^2 f/\partial x^2 - (1/v^2)(\partial^2 f/\partial t^2) = 0$$

As seen, this equation includes only the second derivative with respect to the single space coordinate, $\partial^2 f/\partial x^2$. rather than the sum of second derivatives with respect to all space coordinates:

$$\nabla^2 f = \partial^2 f/\partial x^2 + \partial^2 f/\partial y^2 + \partial^2 f/\partial z^2$$

The solutions to wave equations (10.9) and (10.10) are given as identities

$$\varphi = \int [\rho]\, dV/4\pi\varepsilon_0 R, \quad \mathbf{A} = \mu_0 \int ([\mathbf{J}]/4\pi R) dV \tag{10.11}$$

---

[1] Other gauge conditions are also possible. Thus the condition

$$\nabla\mathbf{A} = -(\mu\varepsilon/c^2)(\partial\varphi/\partial t) - \mu\mu_0\sigma\varphi$$

is convenient to use in the calculation of a field in a conducting medium ($\sigma \neq 0$). Given such a condition, we can abandon (10.9) and (10.10) and use instead

$$\nabla^2\varphi - (\varepsilon\mu/c^2)(\partial^2\varphi/\partial t^2) = \mu\mu_0\sigma(\partial\varphi/\partial t) \tag{10.9a}$$

$$\nabla^2\mathbf{A} - (\varepsilon\mu/c^2)(\partial^2\mathbf{A}/\partial t^2) = \mu\mu_0\sigma(\partial\mathbf{A}/\partial t) \tag{10.10a}$$

The charge density ρ and current density $\mathbf{J}$ under the integral are put in brackets. This means that when evaluating the integral of the potential at any point at an instant t, we should take the charges and currents occurring at a much earlier moment, namely, at the instant

$$t - R/c$$

when

$$[\rho] = \rho(t - R/c), \ [\mathbf{J}] = \mathbf{J}(t - R/c)$$

This condition is in agreement with the notion of a finite velocity c at which an electromagnetic disturbance propagates in a medium: any change in the charge (or current) a distance R remote from the observation point can be detected only after the lapse of a time, R/c, required for the wave to cover the path.

The potentials specified by (10.11) are known as *delayed*, or *retarded potentials*.

For a point charge $q = \rho dV$ or current $i\mathbf{l} = \mathbf{J}dV$ along a conductor length, the expressions for delayed potentials simplify to the form

$$\varphi = [q]/4\pi\varepsilon_0 R$$
$$\mathbf{A} = \mu_0[i\mathbf{l}]/4\pi R \tag{10.12}$$

Deriving equations for delayed potentials. Assume the potential φ is set up by a charge q located at the origin of coordinates and ρ is nonzero only in a small region dV bounding the point 0. Suppose the charge varies with time, $q = q(t)$. The potential φ must then naturally be a function of time too.

To find φ as a function of space coordinates and time, we must solve Eq. (10.9). For the entire region free from the charge (for the entire volume apart from dV), this equation can be written in the form of Riemann's equation

$$\nabla^2\varphi = (1/c^2)(\partial^2\varphi/\partial t^2) \tag{10.13}$$

Proceeding from the considerations of symmetry, it is natural to conclude that φ can depend on the single space coordinate R in the spherical coordinate system. On differentiating, the expression for $\nabla^2\varphi$ then assumes the form

$$\nabla^2\varphi = (1/R^2)(\partial/\partial R)\,[R^2(\partial\varphi/\partial R)] = (1/R)\,[\partial^2(R\varphi)/\partial R^2]$$

where the right side is written in the form convenient for further transformation. Multiplying both sides of (10.13) by R and introducing R under the sign of differentiation with respect to t, we obtain

$$\partial^2(R\varphi)/\partial R^2 - (1/c^2)\,[\partial^2(R\varphi)/\partial t^2] = 0 \tag{10.14}$$

The solution to this equation, found by D'Alembert, is well known

$$R\varphi = f_1(t - R/c) + f_2(t + R/c) \tag{10.15}$$

where f_1 and f_2 are any functions satisfying certain concrete conditions of the problem of interest.

That expression (10.15) indeed satisfies (10.14) is easy to prove by substitution. Of course, the assumption taken here is that the functions f_1 and f_2 have uniquely definite second derivatives. A feature peculiar to f_1 and f_2 lies in that they depend on R and t taken not separately but as a difference $t - R/c$ or sum $t + R/c$. This implies that the first function $f_1(t - R/c)$ represents a wave propagating from the

origin of coordinates in the radial direction with a velocity c [see also Sec. 8.1].

As is apparent from the above analysis, the second function $f_2(t + R/c)$ represents a wave moving along the radius toward the center with a velocity c. From simple physical considerations we can conclude that the second term in (10.15) should be omitted, for it does not conform to the conditions under examination: the varying charge lies at the origin, and, hence, a change in the potential (times the radius) that stems from the change in the magnitude of the charge must first occur at points less remote from the origin and then, with the lapse of time, at more remote points (and not at all in the reverse order). So, setting $f_2 = 0$, we obtain a solution to Eq. (10.13), expressed as

$$\varphi = (1/R)f_1(t - R/c) \tag{10.16}$$

in which the form of function f_1 still remains unknown.

To define this function, one can take the following course of reasoning. If $c \to \infty$, Eq. (10.16) becomes

$$\varphi = (1/R)f_1(t) \tag{10.17}$$

and must represent the solution to Eq. (10.13) when its right side goes to zero, i.e. the solution to Laplace's equation $\nabla^2\varphi = 0$. This solution is well known in electrostatics:

$$\varphi = q\,(t)/4\pi\varepsilon_0 R$$

It remains valid for $q(t) = \rho(t)dV$ when $c \to \infty$ or $R/c \to 0$.

More exactly, when $R/c \ll T$, where T is the period of the sinusoidal changes of the charge in question.

Comparing the latter equality with (10.17), we can specify the sought-for form of the function

$$f_1(t) = q(t)/4\pi\varepsilon_0 = \rho(t)dV/4\pi\varepsilon_0$$

Note that by letting c go to infinity, we have changed only the argument rather than the form of the function, which is now known. Returning to Eq. (10.16), we can rewrite it considering that now we know not only the argument but also the function:

$$\varphi = q(t - R/c)/4\pi\varepsilon_0 R = \rho(t - R/c)dV/4\pi\varepsilon_0 R \tag{10.18}$$

Passing to the integral taken as a sum of potentials built up by individual charges ρdV, we obtain expression (10.11) for a delayed potential. Note that ρ is a function of both t and coordinates x, y, and z, i.e. $\rho = \rho(x, y, z, t)$. The quantity $[\rho]$ under the integral in (10.11) can now be written in an expanded form as

$$[\rho] = \rho(x, y, z, t - R/c)$$

Applying quite similar reasonings, from (10.2) and (10.10) we derive (10.11) for the delayed vector potential **A**.

We shall not dwell further on the general solution, and turn instead to some particular cases.

10.3. Electric Dipole Radiation

Let a wire, with its elemental length expressed by a vector **l** (Fig. 10.2), carry an alternating current $i(t)$. Consider the electromagnetic field produced by the current,

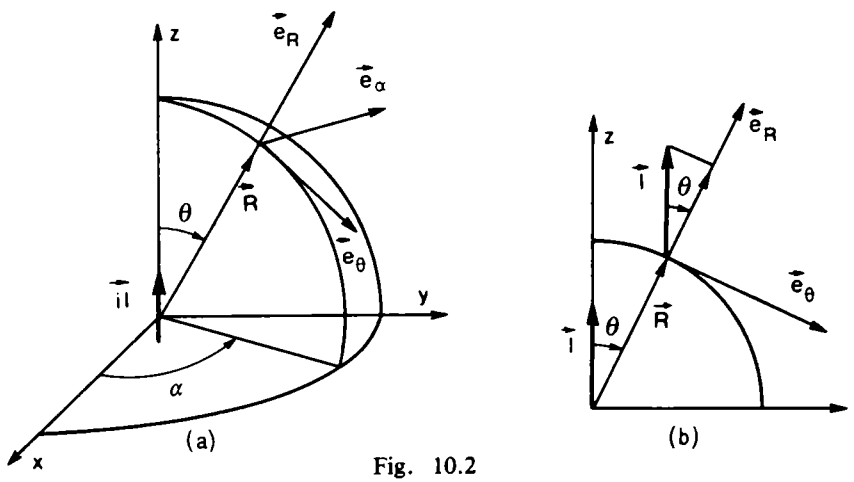

Fig. 10.2

allowing for the finite velocity of field propagation by introducing delayed potentials in the calculation. Assume the current is purely harmonic. Its expression in complex form is

$$\dot{I} \sim I_m \sin\omega t \tag{10.19}$$

It is of advantage to choose such a calculation procedure: to determine first the vector potential $\dot{\mathbf{A}}$, then, after taking its curl, find the magnetic induction vector

$$\text{curl } \dot{\mathbf{A}} = \dot{\mathbf{B}}$$

and, at last, evaluate the electric field strength using Maxwell's first equation

$$\text{curl } \dot{\mathbf{B}} = \mu_0 \varepsilon_0 j\omega \dot{\mathbf{E}} = (j\omega/c^2)\dot{\mathbf{E}}$$

or

$$\dot{\mathbf{E}} = (c^2/j\omega)\,\text{curl } \dot{\mathbf{B}} \tag{10.20}$$

In performing all the calculations it is expedient to make use of the spherical coordinate system (Fig. 10.2*a*). Write out those expressions which will be of avail here:

(1) in the chosen coordinate system (Fig. 10.2*b*),

$$\mathbf{I} = I(\cos\theta \mathbf{e}_R - \sin\theta \mathbf{e}_\theta) \tag{10.21}$$

(2) the only operation met with in the chosen calculation procedure is the curl of **F** in spherical coordinates:

$$\text{curl } \mathbf{F} = \begin{vmatrix} \mathbf{e}_R/R^2\sin\theta & \mathbf{e}_\theta/R\sin\theta & \mathbf{e}_\alpha/R \\ \partial/\partial R & \partial/\partial\theta & \partial/\partial\alpha \\ F_R & RF_\theta & R\sin\theta F_\alpha \end{vmatrix} \tag{10.22}$$

Vector potential. At a point distant R from current element $\dot{I}l$, the potential according to (2.62b) is given by

$$\dot{\mathbf{A}} = (\mu_0/4\pi)(\dot{I}l/R) = \mu_0 \dot{I}l\,[(\cos\theta \mathbf{e}_R - \sin\theta \mathbf{e}_\theta)/4\pi R] \tag{10.23}$$

To take account of the delay of potentials, we should pass from t to $t - R/c$. In other words, for a harmonic function, we should write $\sin\omega(t - R/c)$ in place of $\sin\omega t$. This corresponds to the multiplication of $\dot{I}$ from (10.19) by $e^{-j\omega R/c} = e^{-jkR}$. In this case,

$$\dot{I}e^{-jkR} \sim I_m \sin(\omega t - kR) \tag{10.24}$$

where, as before,

$$k = \omega/c \tag{10.25}$$

Thus, to allow for the delay of a vector potential, we should introduce only the factor e^{-jkR} into expression (10.23) in complex form. The vector potential[1] can then be written as

$$\dot{\mathbf{A}} = \mu_0 \dot{I}l\,[(\cos\theta \mathbf{e}_R - \sin\theta \mathbf{e}_\theta)/4\pi R]e^{-jkR} \tag{10.26}$$

Magnetic field. Determine the vector of magnetic induction from curl $\dot{\mathbf{A}}$ by applying (10.22) to expression (10.26). Since $\dot{\mathbf{A}}$ does not depend on the meridian angle α and does not include the αth component ($\partial/\partial\alpha = 0$, $\dot{A}_\alpha = 0$), the operation curl $\dot{\mathbf{A}}$ simplifies considerably and leads to the equality

$$\dot{\mathbf{B}} = (1/R)\begin{vmatrix} \partial/\partial R & \partial/\partial\theta \\ \dot{A}_R & R\dot{A}_\theta \end{vmatrix} \mathbf{e}_\alpha \tag{10.27}$$

or

$$\begin{aligned} \dot{B} = \dot{B}_\alpha &= (\mu_0 \dot{I}l/4\pi R)\,[(\partial/\partial R)(-\sin\theta e^{-jkR}) - (\partial/\partial\theta)\,[(\cos\theta/R)e^{-jkR}]] \\ &= (\mu_0 \dot{I}l/4\pi)(j\omega/Rc + 1/R^2)\sin\theta e^{-jkR} \end{aligned} \tag{10.28}$$

Under quasistatic conditions at which

$$kR = \omega R/c = 2\pi R/\lambda \ll 1 \tag{10.29}$$

(for the near region) we can neglect the first factor in the parentheses and set $e^{-jkR} \approx 1$. Then, the magnetic field

$$\dot{B} = \dot{B}_\alpha \approx (\mu_0 \dot{I}l/4\pi R^2)\sin\theta \tag{10.30}$$

exactly follows Biot-Savart's law.

On the other hand, for large distances and high frequencies, at which

$$kR = 2\pi R/\lambda \gg 1 \tag{10.31}$$

(for the far region), we can neglect the second factor in the parentheses and assume

---

[1] Here and elsewhere below, the potentials under consideration are delayed potentials.

that

$$\dot{B} = \dot{B}_\alpha = \mu_0 \dot{I} l (j\omega/4\pi Rc) \sin\theta e^{-jkR} \tag{10.32}$$

This is exactly the magnetic field of the electromagnetic wave being radiated.

A remarkable thing is that the radiation field increases with frequency and decreases inversely with the first power of the radius, i.e. the field decays much more weakly with distance than Biot-Savart's component (10.30).

Considering the relation between the wavelength of a free wave and frequency, $\lambda = c/f = 2\pi c/\omega$, expression (10.32) can be written in the form

$$\dot{B} = \dot{B}_\alpha = (\mu_0 \dot{I} l/2)(j/\lambda R) \sin\theta e^{-jkR} \tag{10.33}$$

The magnetic field is oriented tangent to the circles of latitude because $\dot{B} = \dot{B}_\alpha$ (see Figs. 10.1 and 10.2) and its strength decreases from the equator to the poles ($\sin\theta$).

Electric field. The $\dot{E}$ vector is found by Maxwell's first equation (10.20). Using (10.22) at $B = B_\alpha$ given by (10.28) and at $\partial/\partial\alpha = 0$, we obtain

$$\operatorname{curl} \dot{\mathbf{B}} = \begin{vmatrix} \mathbf{e}_R/R^2 \sin\theta & \mathbf{e}_\theta/R \sin\theta & \mathbf{e}_\alpha/R \\ \partial/\partial R & \partial/\partial\theta & 0 \\ 0 & 0 & R\sin\theta\dot{B}_\alpha \end{vmatrix}$$

Performing the differentiation, we find that

$$\begin{aligned} \operatorname{curl} \dot{\mathbf{B}} = (\mu_0 \dot{I} l e^{-jkR}/4\pi R) \{ & (j\omega/Rc \\ & + 1/R^2) 2\cos\theta\, \mathbf{e}_R \\ & + [(j\omega/c)^2 + j\omega/Rc + 1/R^2] \sin\theta \mathbf{e}_\theta \} \end{aligned} \tag{10.34}$$

Keeping in mind this expression, from (10.20) we get

$$\begin{aligned} \dot{\mathbf{E}} = (j\omega\mu_0 \dot{I} l e^{-jkR}/4\pi R) \{ & [c/j\omega R \\ & + (c/j\omega R)^2 2]\cos\theta \mathbf{e}_R + [1 + c/j\omega R \\ & + (c/j\omega R)^2] \sin\theta \cdot \mathbf{e}_\theta \} \end{aligned} \tag{10.35}$$

This expression for the electric field also contains a factor that decays inversely as the radius. For the far region, we can disregard all other factors as against this factor, and so the expression reduces to the form

$$\dot{E} = \dot{E}_\theta = (j\omega\mu_0 \dot{I} l/4\pi R) \sin\theta e^{-jkR} \tag{10.36a}$$

or

$$\dot{\mathbf{E}} = (-j\omega\mu_0/4\pi R^2) \dot{I} e^{-jkR} \mathbf{l} \times \mathbf{R} \tag{10.36b}$$

since $\mathbf{l} \times \mathbf{R} = -lR \sin\theta \cdot \mathbf{e}_\theta$.

In the far region, the components **B** and **E** are orthogonal (see Fig. 10.1) and related to each other by the factor c:

$$\dot{E} = \dot{E}_\theta = c\dot{B} = c\dot{B}_\alpha \tag{10.37}$$

Radiation power. This power can be found by evaluating the Poynting flux through the closed surface.

The Poynting vector in the far region specified by (10.32) and (10.37) takes the form

$$\underline{\Pi} = (1/\mu_0)\,\dot{\mathbf{E}} \cdot \overset{*}{\mathbf{B}} = (c/\mu_0)\,B_\alpha^2 \mathbf{e}_R$$
$$= (\mu_0/c)(\omega Il/4\pi R)^2 \sin^2\theta \cdot \mathbf{e}_R = \Pi_R \mathbf{e}_R \tag{10.38}$$

The Poynting vector is real here, which shows that at any moment the field carries away the source energy out of the far region.

After integrating the real vector over the sphere, we obtain the real quantity (active power) which represents the entire radiation power:

$$P = \oint_\pi \Pi_R dS$$
$$= \int_{\theta=0} (\mu_0/c)\,(\omega Il/4\pi R)^2 \sin^2\theta\, 2\pi\, R^2 \sin\theta\, d\theta$$
$$= (\mu_0/c)\,[(\omega^2/6\pi)(Il)^2 \tag{10.39}$$

Here, we have taken into account that $dS = 2\pi R \sin\theta\, Rd\theta$ is the spherical surface element in the form of a belt between two latitudinal circles differing in angle by $d\theta$; $R \sin\theta$ is the radius of a latitudinal circle with an angle θ reckoned from the "north pole".

Considering the electric dipole (see Fig. 10.1) with a varying electric moment $\dot{q}\mathbf{l}$, we must assume that the wire connecting the two charged spheres carries a current $\dot{I} = j\omega\dot{q}$. In this case, the radiation of an element $\dot{I}\mathbf{l}$ is identical to the radiation of the electric dipole having a moment

$$\dot{q}\mathbf{l} = I\mathbf{l}/j\omega \tag{10.40}$$

In all expressions for the radiation field and power we can then set $I\mathbf{l} = j\omega q\mathbf{l}$. Hence, we have

$$\dot{B} = \dot{B}_\alpha = -[\mu_0\omega^2/4\pi Rc]\,\dot{q}\mathbf{l} \sin\theta \cdot e^{-jkR} \tag{10.41}$$

$$\dot{E} = \dot{E}_\theta = c\dot{B}_\alpha \tag{10.42}$$

and

$$P = (\mu_0/c)(\omega^4/6\pi)(ql)^2 \tag{10.43}$$

Input resistance of a radiating dipole — radiation resistance. This is a very important characteristic for a supply source sustaining charge (and current) oscillations in the dipole. The characteristic can also be complex in character, i.e. can include a reactive component. The reactive component of the impedance corresponds to reversible changes in the energy of the field near the dipole. This component must be offset by inserting the appropriate inductances or capacitances. The active component is an equivalent input resistance r_{in} responsible for $I^2 r_{in}$ losses, i.e. dissipation of the energy as heat in accordance with Joule's law. From (10.39) it is obvious that

$$r_{in} = P/I^2 = (\mu_0/c)(\omega^2/6\pi)l^2 = 80\pi^2(l^2/\lambda^2), \text{ ohms} \tag{10.44}$$

Thus, at a wavelength of 100 m and a dipole length of 10 m, r_{in} is near 7.9 ohms.

10.4. Half-Wave Antenna

At high frequencies, one cannot sometimes consider a radiating wire length l to be very small as against the wavelength. Where this is the case, the current in all the portions of the wire cannot be taken identical. Proceeding from the theory of long lines, it can readily be shown that the current of a finite value at the input end of the line reduces to zero at its open end (the voltage at the open end is correspondingly at a maximum). If the length of each of the two wire pieces connected to a source is close to one-fourth the wavelength, $\lambda/4$, then standing waves appear in these wires, with the current distributed almost sinusoidally along each wire piece. Fig. 10.3*a* shows the distribution of current and linear charge density, τ. Fig. 10.3*b* illustrates the antenna and its image in the conducting ground. The vertical wire and its image form a half-wave antenna.

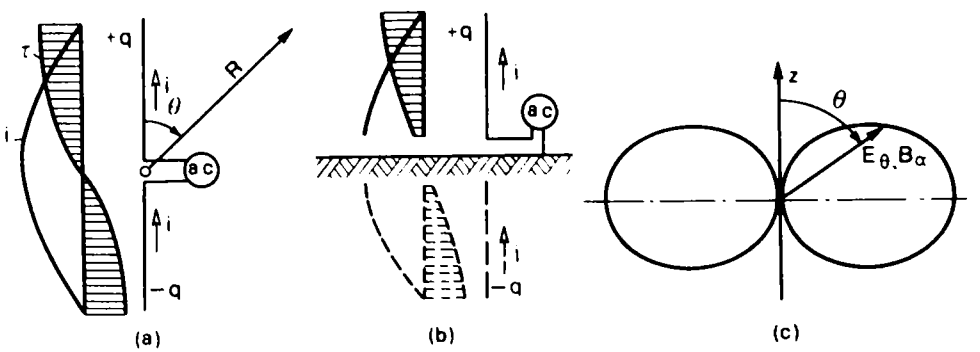

Fig. 10.3

The distribution of current along the wire can be taken cosinusoidal:

$$\dot{I} = \dot{I}_0 \cos(2\pi l/\lambda)$$

where l is the distance from the center of the antenna to an observation point. The field for each element of the antenna can be found from the formulas derived above. For this, we have to replace $\dot{I}$ by $\dot{I}dl$ and take the integral along l between the limits $-\lambda/4$ and $+\lambda/4$. The distances R (Fig. 10.4) from various points of the antenna to the observation point are different. As seen from Fig. 10.4, if $R = R_0 - l\cos\theta$ for a far region, we can assume to a high degree of accuracy that $l\cos\theta$ is negligible relative to R_0. But this subtrahend can be appreciable compared with the wavelength λ (recall that the entire length of the antenna is equal to $\lambda/2$). If this is the case, the difference in phase between the waves transmitted from various points of the antenna can also prove considerable.

Refer now to (10.32). Let us replace l by dl and assume that $R \approx R_0$ and the angle θ is identical for the radii R and R_0 (see Fig. 10.4). But we will consider the phase factor

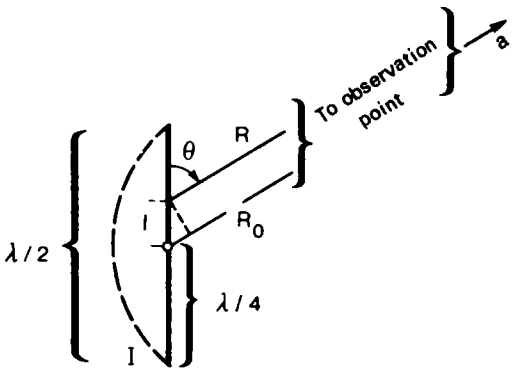

Fig. 10.4

to be dependent on the location of a radiation point at the antenna (i.e. dependent on the coordinate l):

$$e^{-jkR} = e^{-jkR_0} e^{jkl\cos\theta} \tag{10.45}$$

To determine the total field, we should take the sum over all components of the field radiated by the elements of the antenna along its entire length; in other words, we should take the integral along l between $-\lambda/4$ and $+\lambda/4$:

$$\dot{B} = \dot{B}_0 e^{-jkR_0} \int_{-\lambda/4}^{+\lambda/4} e^{-jkl\cos\theta} \cos kl \, dl$$

where $\dot{B}_0 = \mu_0 \dot{I}_0 j\omega \sin\theta / 4\pi Rc$.

Upon integrating, we obtain

$$\dot{B} = \dot{B}_\alpha = j(\mu_0 \dot{I}_0 / 2\pi R \sin\theta) e^{-jkR_0} \cos[(\pi/2)\cos\theta] \tag{10.46}$$

In the far region (for which we have found the solution) the electric field vector is normal to the **B** vector and differs from the latter by the factor c:

$$\dot{E} = \dot{E}_\theta = c\dot{B} = c\dot{B}_\alpha \tag{10.47}$$

10.5. Directional Radiation

Directivity is of exceptional importance in many cases of radio signal transmission. In the radiators discussed above, a certain directional effect is also present. Thus, the dependence of B_α and E_θ on the angle θ is evident from formula (10.46) and Fig. 10.3*c*. But the above systems radiate energy that propagates within a very wide solid angle. Therefore, in the absence of energy absorption, the radiation power decays inversely with the square of the distance, and each of the electromagnetic field components decreases inversely with the radius. The smaller the angular width of the beam of energy being radiated, the larger the distance for which it is possible to radiate a readable signal.

One of the simplest systems used to effect the desired directivity consists of a number of half-wave antennae spaced at quite certain distances one from the other; provision is made within the system to keep currents strictly in phase or maintain a

definite phase shift specified by the calculations. The total field is found as the sum of all field components radiated by the antennae. For a far region, the distances to each of the antennae may practically be taken equal. However, a small difference in distances has a substantial effect on the phase displacement of waves traveling from individual antennae of the entire system, and so this effect must be taken into account.

Consider, for example, five half-wave antennae arranged parallel to each other and spaced at $\lambda/2$ intervals. Fig. 10.5 displays this antenna array depicted so that the plane of the drawing is normal to the antennae and passes through the center of each. From the figure it is seen that the distance from each consecutive antenna (in the ascending order of its ordinal number) to the observation point decreases by $b\cos\alpha$, where b is the distance between the neighboring antennae equal to $\lambda/2$, and α is the meridian (direction) angle, i.e. the angle between the antenna base line and the radius drawn to the observation point.

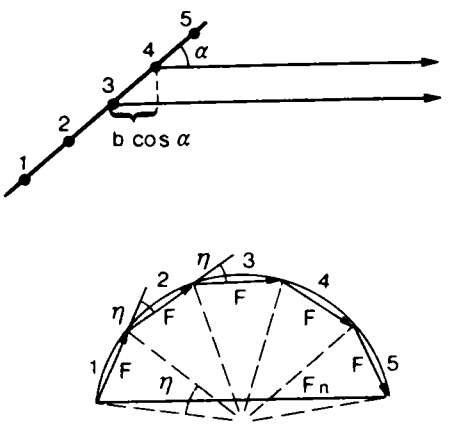

Fig. 10.5

The field of each antenna differs in phase from the field of its neighbor by the angle

$$\eta = (2\pi b/\lambda)\cos\alpha = \pi\cos\alpha$$

By summing up the five field vectors (see Fig. 10.5), considering that each has a length F, we obtain

$$F_5 = F[\sin(5\eta/2)/\sin(\eta/2)]$$

In detemining the vector $\dot{B}_\alpha$, we must take (10.46) as the expression for F. Multiplying $\dot{B}_\alpha$ by c gives the expression for $\dot{E} = \dot{E}_\theta$ in the far region.

The directional radiation pattern appears in Fig. 10.6 (for plane $\theta = \pi/2$). The dependence on θ is the same as that presented in the pattern of Fig. 10.3*c*.

Another method of achieving the directional effect is to dispose the radiator in the focus of a parabolic "mirror" (an array of conductors). The diameter of such mirrors reaches dozens of meters. The radiators described above find use as radio beacons and radiation sources in radiolocation. Parabolic systems also enjoy wide use as

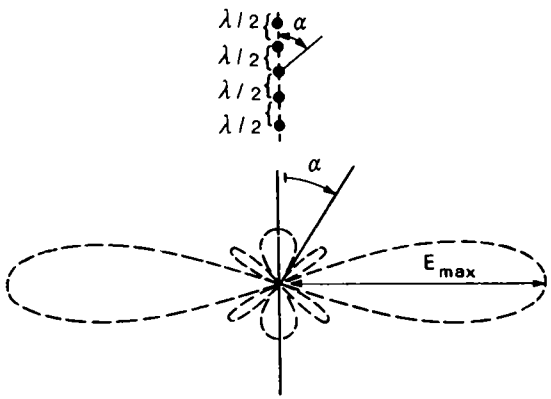

Fig. 10.6

receiving antennae in radio astronomy. Lasers display exceptional directivity. They concentrate radiation into a narrow beam that practically has a parallel pattern.

10.6. Summary

Hertzian vector. Considering the radiation of an electric dipole

$$\dot{\mathbf{p}} = \dot{q}\mathbf{l}$$

it often proves useful to introduce one more type of vector potential

$$\dot{\mathbf{Z}} = \dot{\mathbf{p}}e^{-jkR}/4\pi\varepsilon_0 R \tag{10.48}$$

It is known as *the Hertzian vector or polarization potential.* This vector is equal to the dipole moment divided by the distance R from the dipole to an observation point (accurate to a constant factor). Where the dipole moment varies with time, its value is taken at an instant $t - R/c$ to allow for the finite velocity c of the polarization potential. For a sinusoidally varying potential given by the complex number $\dot{\mathbf{p}}$, the delay is expressed by the factor e^{-jkR} (the delay in phase by kR).

The $\dot{\mathbf{Z}}$ vector shows a number of remarkable features. First of all, it satisfies the wave equation which will now be written in complex form as

$$\nabla^2\dot{\mathbf{Z}} + (\omega/c)^2\dot{\mathbf{Z}} = 0 \tag{10.49}$$

This equation differs from (10.10) in that it represents the time differentiation, $\partial/\partial t$, of the given harmonic time function $\mathbf{Z}(t)$ as the product of $j\omega$ and the complex quantity $\dot{\mathbf{Z}}$.

The validity of (10.49) follows from the fact that an analogous equation is applicable to the vector potential [see (10.12) and (10.10)]

$$\dot{\mathbf{A}} = (\mu_0\dot{\mathbf{I}}l/4\pi R)e^{-jkR} = (j\omega\dot{q}\mathbf{l}/4\pi\varepsilon_0 c^2 R)e^{-jkR} \tag{10.50}$$

This expression is identical to (10.12) for a harmonic function given in complex form. Indeed, (10.48) and (10.50) represent the identical functions of time and coordinates

$$\dot{C}e^{-jkR}/R$$

and only differ by the value of constant factor $\dot{C}$ = constant.

We now consider two more features of the Hertzian vector. Both a scalar potential $\dot{\varphi}$ and vector potential $\dot{\mathbf{A}}$ can be expressed in terms of the Hertzian vector:

$$\dot{\varphi} = -\operatorname{div} \dot{\mathbf{Z}} \text{ and } \dot{\mathbf{A}} = j\omega\dot{\mathbf{Z}}/c^2 \tag{10.51}$$

To support the point, we can perform the divergence operation in the spherical coordinate system and compare the solution with (10.23) for $\dot{\mathbf{A}}$.

Substituting(10.51) for $\dot{\varphi}$ and $\dot{\mathbf{A}}$ into the expression for the total electric field

$$\dot{\mathbf{E}} = -\operatorname{grad}\dot{\varphi} - j\omega\dot{\mathbf{A}}$$

we find that

$$\dot{\mathbf{E}} = \operatorname{grad}\operatorname{div}\dot{\mathbf{Z}} - (j\omega/c)^2\dot{\mathbf{Z}}$$

or, evaluating grad div $\dot{\mathbf{Z}}$ by use of the known identity, we obtain

$$\dot{\mathbf{E}} = \nabla^2\dot{\mathbf{Z}} + \operatorname{curl}\operatorname{curl}\dot{\mathbf{Z}} - (j\omega/c)^2\dot{\mathbf{Z}}$$

At last, considering (10.49), the expression becomes

$$\dot{\mathbf{E}} = \operatorname{curl}\operatorname{curl}\dot{\mathbf{Z}} \tag{10.52}$$

The expression above describes the total electric field of the dipole expressed as the Hertzian vector $\dot{\mathbf{Z}}$.

The magnetic field vector can be found from (10.51) for $\dot{\mathbf{A}}$:

$$\dot{\mathbf{B}} = \operatorname{curl}\dot{\mathbf{A}} = (j\omega/c^2)\operatorname{curl}\dot{\mathbf{Z}} \tag{10.53}$$

In the case of one dipole, the use of the Hertzian vector gives the known results and in essence does not offer any advantages in calculations. But its application in solving a number of design problems can result in mathematical simplifications.

The Hertzian vector can also be found for a field due to an alternating current $\dot{I}$ flowing in conductors l. In the elemental conductor $d\mathbf{l}$, the product $\dot{I}d\mathbf{l}$ can readily be expressed in terms of an infinitesimal electric dipole:

$$\dot{I}d\mathbf{l} = j\omega\dot{q}d\mathbf{l} = j\omega d\dot{\mathbf{p}} \tag{10.54}$$

The Hertzian vector corresponding to the current $\dot{I}(l)$ in the conductor is then defined as an integral:

$$\dot{\mathbf{Z}} = (1/j\omega)\int(\dot{I}/4\pi\varepsilon_0 R)e^{-jkR}d\mathbf{l} \tag{10.55}$$

Applying this integral to the closed loop, where $\dot{I}$ is the same in all the loop elements, we can take $\dot{I}$ outside the integral. For this loop,

$$\dot{\mathbf{Z}} = (\dot{I}/j\omega)\oint(\dot{I}/4\pi\varepsilon_0 R)e^{-jkR}d\mathbf{l} \tag{10.56}$$

Using this calculation method, we can determine the radiation of an antenna in the form of a loop carrying current $\dot{I}$ (Fig. 10.7).

Magnetic dipole. For the region $R \gg d$, where d represents the linear dimensions of area S bounded by the loop carrying current $\dot{I}$, a loop antenna can be represented as a magnetic dipole with a magnetic moment

$$\dot{\mathbf{m}} = \dot{I}\mathbf{S} \tag{10.57}$$

Proceeding from the formal analogy between magnetostatics and electrostatics,

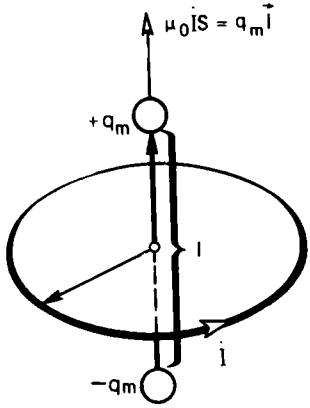

Fig. 10.7

we can represent the magnetic moment as two magnetic charges q_m opposite in sign located at the ends of length pieces $\mathbf{l}$ (see Fig. 10.7):

$$\mu_0 \dot{\mathbf{m}} = \dot{q}_m \mathbf{l} \tag{10.58}$$

We can extend this analogy and obtain the Hertzian magnetic vector or magnetic polarization potential

$$\dot{\mathbf{Z}}_m = (\dot{\mathbf{m}}/4\pi R) e^{-jkR} \tag{10.59}$$

From the symmetry of Maxwell's equations

$$\operatorname{curl} \dot{\mathbf{H}} = j\omega\varepsilon_0 \dot{\mathbf{E}}, \operatorname{curl} \dot{\mathbf{E}} = -j\omega\mu_0 \dot{\mathbf{H}}$$

it is possible to find (up to a sign!) that the following quantities are formally analogous:

$$\dot{\mathbf{E}} \sim \dot{\mathbf{H}}, \dot{\mathbf{H}} \sim \dot{\mathbf{E}}, j\omega\varepsilon_0 \sim -j\omega\mu_0 \tag{10.60}$$

Denoting the $\dot{\mathbf{Z}}$ vector discussed earlier as a Hertzian electric vector $\dot{\mathbf{Z}}_e$ enables us to compare the equations expressed in terms of $\dot{\mathbf{Z}}_e$ with analogous equations expressed in terms of $\dot{\mathbf{Z}}_m$, keeping certainly in mind (10.60):

$\dot{\mathbf{Z}}_e = \dot{\mathbf{p}} e^{-jkR}/4\pi\varepsilon_0 R$	$\dot{\mathbf{Z}}_m = \mu_0 \dot{\mathbf{m}} e^{-jkR}/4\pi R$
$\dot{\mathbf{E}} = \operatorname{curl}\operatorname{curl} \dot{\mathbf{Z}}_e$	$\dot{\mathbf{H}} = \operatorname{curl}\operatorname{curl} \dot{\mathbf{Z}}_m$
$\dot{\mathbf{H}} = j\omega\varepsilon_0 \operatorname{curl} \dot{\mathbf{Z}}_e$	$\dot{\mathbf{E}} = -j\omega\mu_0 \operatorname{curl} \dot{\mathbf{Z}}_m$
$\dot{\mathbf{p}} = \dot{q}\mathbf{l}$	$\dot{\mathbf{m}} = \dot{I}\mathbf{S}$
$d\dot{\mathbf{p}} = \dot{\mathbf{P}} dV$	$d\dot{\mathbf{m}} = \dot{\mathbf{M}} dV$

In the latter line, the electric moment $d\dot{\mathbf{p}}$ of elemental volume dV is expressed as the polarization of a substance. Similarly, the magnetic moment is expressed as the magnetization. Substituting the desired quantities into the Hertzian vector expression, we obtain $d\mathbf{Z}$. By integrating it, we can evaluate $\mathbf{Z}_e$ and $\mathbf{Z}_m$.

The expression for $\dot{\mathbf{H}}$ in the left column is derived from (10.53) for $\dot{\mathbf{B}}$ at $\dot{\mathbf{H}} = \dot{\mathbf{B}}/\mu_0$ and $c^2 = 1/\mu_0\varepsilon_0$.

All the expressions given here hold for a homogeneous medium: $\mu = \varepsilon = 1$ and $\sigma = 0$. They can be extended to cover any medium on replacing μ_0 and ε_0 by $\mu_0\mu$ and $\varepsilon_0\varepsilon$ and vacuum light velocity c by the complex quantity $c/\sqrt{\varepsilon\mu}$. Obviously, the factor $\exp(-jkR\sqrt{\varepsilon\mu})$ allows here for attenuation as well as decaying.

The above equations are also valid for two different media. Basic equations are used to describe each medium and then the solutions are found to satisfy the boundary conditions at the interface between the media.

Sommerfeld applied exactly this approach to solve a very important problem on the propagation of radio waves over the earth's surface. It is of interest to note that the Sommerfeld method for calculating the Hertzian vector is in extensive use in geological survey. One of the prospecting methods uses a multiturn ring dozens of meters in diameter, which is laid on the ground surface and then an alternating current of a very low frequency (about 1 Hz or even tenths of a hertz) is passed through it. At a large distance from the ring, the field at the ground surface is measured. This field varies with ground conductivity. Its pattern of variation is indicative of the ground structure composed of layers and inclusions of different conductivities.

Reciprocity principle and receiving antennae. If in the far region of a wave radiating element of infinitesimal length $d\mathbf{l}_1$ there is a second vanishingly small element $d\mathbf{l}_2$, the first element will induce an emf in the second:

$$d\dot{\mathscr{E}} = d\dot{\mathbf{E}}d\mathbf{l}_2$$

Expressing $d\dot{\mathbf{E}}$ as given by (10.36b), we obtain

$$d\dot{\mathscr{E}} = -(j\omega\mu_0/4\pi R^2)\dot{I}e^{-jkR}(d\mathbf{l}_1 \times \mathbf{R})d\mathbf{l}_2 \qquad (10.61)$$

Taking $d\mathscr{E}$ as a quantity proportional to the vector product $(d\mathbf{l}_1 \times \mathbf{R})d\mathbf{l}_2$, expressions (10.61) can be written in the form

$$d\dot{\mathscr{E}} = (j\omega\mu_0/4\pi R^2)\dot{I}e^{-jkR}(d\mathbf{l}_2 \times \mathbf{R})d\mathbf{l}_1$$

where $\mathbf{R}$ is the vector drawn from $d\mathbf{l}_1$ into $d\mathbf{l}_2$.

From the latter expression, which is symmetric with respect to $d\mathbf{l}_1$ and $d\mathbf{l}_2$, it follows that on *interchanging the positions of a receiving and a radiating element, one element will induce the same emf $d\mathscr{E}$ in the other at the same current I in the first.* This is the principle of reciprocity for a transmitting and a receiving antenna. The principle holds for electromagnetic waves propagating in vacuum and in any isotropic and nongyrotropic linear medium. Free charges present in a stationary magnetic field (as is the case for the ionosphere) disturb the reciprocity principle.

Integrating (10.61) by lengths $d\mathbf{l}_1$ and $d\mathbf{l}_2$, we can show that the reciprocity theorem is valid not only for elementary oscillators but also for oscillators of a more complex configuration.

Radiation from a moving charge. When a charge q moves with a certain acceleration[1], it radiates electromagnetic waves. The simplest way of calculating the radiation is to use formulas (10.41) and (10.42) for the field of an electric dipole in a far

[1] In an inhomogeneous medium, a charge can radiate waves at a constant velocity too (Čerenkov radiation).

region:

$$\dot{\mathbf{B}} = [\mu_0(j\omega)^2/4\pi Rc]\dot{\mathbf{p}}e^{-jkR} \times \mathbf{e}_R$$

and

$$\dot{\mathbf{E}} = c\dot{\mathbf{B}} \times \mathbf{e}_R$$

These formulas are written on the assumption that the moment varies sinusoidally. The second time derivative is represented by the factor $(j\omega)^2$.

For an arbitrary function of time

$$\mathbf{p} = \mathbf{p}(t) = q\mathbf{l}(t) \tag{10.62}$$

we have

$$\mathbf{B}(t) = (\mu_0/4\pi Rc)q[\partial^2\mathbf{l}(t - R/c)/\partial t^2] \times \mathbf{e}_R \tag{10.63a}$$

$$\mathbf{E}(t) = c\mathbf{B}(t) \times \mathbf{e}_R \tag{10.63b}$$

Considering a moving charge q (Fig. 10.8), it is always safe to assume that a stationary charge $-q$ lies at the point of reference of vector $\mathbf{l}(t)$. The two charges then display an electric dipole moment (10.62). The assumption that a stationary charge $-q$ exists at the point from which the "arm" $\mathbf{l}$ is to be reckoned does not certainly affect the field radiated by a moving charge.

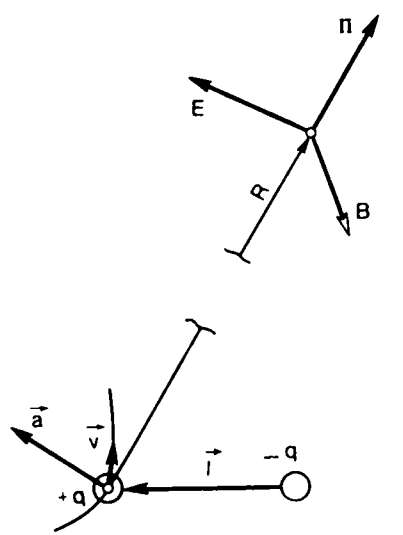

Fig. 10.8

It is obvious that $\partial\mathbf{l}/\partial t = \mathbf{v}(t)$ is the charge velocity, $\partial^2\mathbf{l}/\partial t^2 = \mathbf{a}(t)$ is the acceleration, and the quantities $\mathbf{a}$, $\mathbf{v}$, and $\mathbf{l}$ are vectors. The finite velocity (light velocity) of field propagation, i.e. the delay in propagation, is taken into account by substituting $t - R/c$ for the argument t. Expression (10.63a) can thus be rewritten to take the form

$$\mathbf{B}(t) = (\mu_0/4\pi Rc)q\mathbf{a}(t - R/c) \times \mathbf{e}_R \tag{10.64}$$

The Poynting vector[1] is then given by

$$\Pi = (1/\mu_0)\mathbf{E} \times \mathbf{B} = [1/\varepsilon_0(4\pi c)^2]q^2a^2 \sin^2 \eta \mathbf{e}_R \tag{10.65}$$

where η is the angle between the vectors $\mathbf{a}$ and $\mathbf{e}_R$ (see Fig. 10.8).

Taking the integrtal over the sphere of radius R (10.39) gives the instantaneous value of radiation power:

$$p(t) = \oint \Pi d\mathbf{S} = (1/6\pi\varepsilon_0 c^3)q^2a^2 \tag{10.66}$$

In the case of harmonic oscillations, we should replace a^2 by $(\omega^2 l)^2$, where l is the half of the peak-to-peak value (i.e. the swing) of charge oscillations, and besides, pass to the time-averaged value of power. If l is the amplitude of harmonic oscillations, the rms (effective) value is known to be equal to $1/\sqrt{2}$ of the peak amplitude. The expression for power then assumes the form

$$P = (ql\omega^2)^2/12\pi\varepsilon_0 c^3 \tag{10.67}$$

In this expression, it is possible to replace the frequency $\omega = 2\pi f$ by a wavelength $\lambda = 2\pi c/\omega$ and write

$$P = (4/3)\pi^3 c(ql)^2/\varepsilon_0\lambda^4 \tag{10.68}$$

The radiation power thus varies inversely as the wavelength raised to the fourth power.

Blue sky. Why is the sky blue? Although this question does not certainly relate to the problems in electrical engineering, it might be instructive to note that the above equations of classical electrodynamics (dependence of radiation power on frequency), explain why the sky assumes a blue color.

The electric field of a light wave polarizes air molecules. The dipole moments so formed oscillate at a frequency of light waves, and we can assume to a first approximation that the amplitudes of all molecular dipoles are proportional to the intensity of radiation in the entire spectrum.

So, for white light, the oscillation amplitudes of orbital or molecule electrons (electrons linked to molecules) distribute themselves uniformly over the entire frequency spectrum. But the radiation power given by (10.67) varies directly with the fourth power of frequency. Therefore, the light scattered by molecules appears blue: the atmosphere is just a host of luminaries of all spectral colors, of which the sources of blue give light more intensely than the sources of red.

Molecular generators. Conversion of the electromagnetic energy of one frequency f_1 to that of another frequency f_2 presents an important problem in electrical engineering. The problem is of particular significance in the case where the power given off at a frequency f_2 is higher than the power of electromagnetic waves at a frequency f_1. As noted in Sec. 7.3, this type of energy conversion does not certainly violate the law of conservation of energy. A very important group of generators

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[1] The solution is found after evaluating the quadruple vector product $[(\mathbf{a} \times \mathbf{e}_R) \times \mathbf{e}_R] \times (\mathbf{a} \times \mathbf{e}_R) = \mathbf{e}_R[\mathbf{a}^2 - (\mathbf{a}\mathbf{e}_R)^2] = \mathbf{e}_R a^2(1 - \cos^2\eta)$. All the operations should be performed by using consecutively the known formula $\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{AC}) - \mathbf{C}(\mathbf{AB})$.

(lasers) rely on the laws of quantum physics and physical optics. One of the most distinction features of the beam generated by a laser is its uniformity in phase due to phase coincidence of the waves of all molecular radiators of the laser.

APPENDICES

A1. Vector Algebra

Vector resolution

$$\mathbf{A} = \mathbf{e}_x A_x + \mathbf{e}_y A_y + \mathbf{e}_z A_z$$
$$\mathbf{A} = \mathbf{e}_r A_r + \mathbf{e}_\alpha A_\alpha + \mathbf{e}_z A_z$$
$$\mathbf{A} = \mathbf{e}_R A_R + \mathbf{e}_\theta A_\theta + \mathbf{e}_\alpha A_\alpha$$

Scalar product

$$C = \mathbf{AB} = A_x B_x + A_y B_y + A_z B_z = \mathbf{BA}$$
$$C = AB \cos(\mathbf{A}, \mathbf{B})$$

Vector product

$$\mathbf{C} = \mathbf{A} \times \mathbf{B} = \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} = -\mathbf{B} \times \mathbf{A}$$
$$C = AB \sin(\mathbf{A}, \mathbf{B}), \mathbf{C} \perp \mathbf{A}; \mathbf{C} \perp \mathbf{B}$$

Triple vector product

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{AC}) - \mathbf{C}(\mathbf{AB})$$

A2. Vector Identities

$$\nabla\varphi(u) = (\partial\varphi/\partial u)\nabla u$$
$$\nabla(\varphi\psi) = \psi\nabla\varphi + \varphi\nabla\psi$$
$$\nabla(\mathbf{AB}) = \mathbf{A} \times (\nabla \times \mathbf{B}) + (\mathbf{A}\nabla)\mathbf{B} + \mathbf{B} \times (\nabla \times \mathbf{A}) + (\mathbf{B}\nabla)\mathbf{A}$$

where

$$(\mathbf{A}\nabla) = A_x(\partial/\partial x) + A_y(\partial/\partial y) + A_z(\partial/\partial z)$$
$$\nabla(\varphi\mathbf{F}) = \varphi\nabla\mathbf{F} + \mathbf{F}\nabla\varphi$$
$$\nabla(\mathbf{A} \times \mathbf{B}) = \mathbf{B}(\nabla \times \mathbf{A}) - \mathbf{A}(\nabla \times \mathbf{B})$$
$$\operatorname{div}\operatorname{curl}\mathbf{F} = 0, \operatorname{curl}\operatorname{grad}\varphi = 0$$
$$\nabla \times (\varphi\mathbf{F}) = \varphi\nabla \times \mathbf{F} - \mathbf{F} \times \nabla\varphi$$
$$\nabla \times (\mathbf{A} \times \mathbf{B}) = (\mathbf{B}\nabla)\mathbf{A} - (\mathbf{A}\nabla)\mathbf{B} + \mathbf{A}\nabla\mathbf{B} - \mathbf{B}\nabla\mathbf{A}$$
$$\operatorname{curl}\operatorname{curl}\mathbf{F} = \operatorname{grad}\operatorname{div}\mathbf{F} - \nabla^2\mathbf{F}$$

A3. Integral Identities

1. $\oint \varphi d\mathbf{S} = \int \operatorname{grad}\varphi \cdot dV$
2. $\oint \mathbf{F} d\mathbf{S} = \int \operatorname{div}\mathbf{F} \cdot dV$

3. $\oint \mathbf{F}d\mathbf{l} = \int \operatorname{curl}\mathbf{F} \cdot d\mathbf{S}$

4. $\oint \mathbf{F} \times d\mathbf{S} = -\int \operatorname{curl}\mathbf{F} \cdot dV$

5. $\oint \mathbf{F}(\mathbf{A}d\mathbf{S}) = \int (\mathbf{A}\nabla)\mathbf{F} \cdot dV + \int \mathbf{F} \operatorname{div}\mathbf{A} \cdot dV$

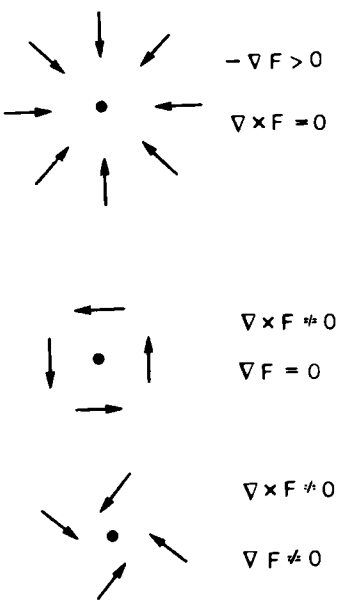
Among these identities, the second is known as the Ostrogradsky-Gauss theorem and the third as Stokes' theorem. The first, second, and fourth identities can serve as statements of basic operations:

6. $\operatorname{grad}\varphi = \nabla\varphi = \lim_{V \to 0} (\oint \varphi d\mathbf{S})/V$

7. $\operatorname{div}\mathbf{F} = \nabla\mathbf{F} = \lim_{V \to 0} (\oint \mathbf{F}d\mathbf{S})/V$

8. $\operatorname{curl}\mathbf{F} = \nabla \times \mathbf{F} = \lim_{V \to 0} (-\oint \mathbf{F} \times d\mathbf{S})/V$

Statements 6, 7, and 8 are noted for uniformity. What is needed for performing one and the same operation is to multiply the operator ∇ by scalar (6), the scalar by vector (7), and last, the vector by vector (8).



The figures above are the sketches Maxwell used in his treatise to illustrate the operations involving the operator ∇, which were not yet popular at the time.

A4. Simple Orthogonal Coordinate Systems

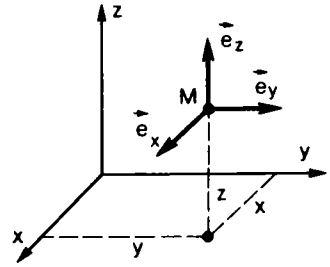


Fig. A1

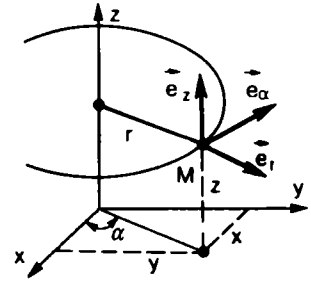


Fig. A2

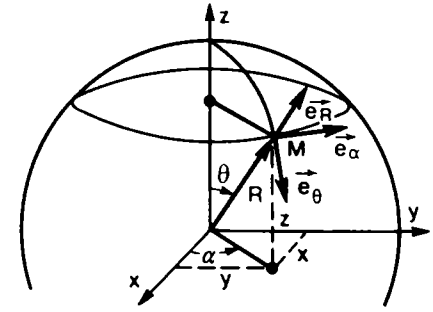


Fig. A3

Coordinates	Unit vectors	Line elements	Relations between coordinate systems
Rectangular, x, y, z (Fig. A1)	$\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z$	dx, dy, dz	—
Cylindrical, r, α, z (Fig. A2)	$\mathbf{e}_r, \mathbf{e}_\alpha, \mathbf{e}_z$	$dr, rd\alpha, dz$	$x = r\cos\alpha, r = \sqrt{x^2 + y^2}$ $y = r\sin\alpha, \tan\alpha = y/x$ $z = z$
Spherical, R, θ, α (Fig. A3)	$\mathbf{e}_R, \mathbf{e}_\theta, \mathbf{e}_\alpha$	$dR, Rd\theta, R\sin\theta\, d\alpha$	$x = R\sin\theta\cos\alpha$ $y = R\sin\theta\sin\alpha$ $z = R\cos\theta$ $R^2 = x^2 + y^2 + z^2$ $\cos\theta = z/\sqrt{x^2 + y^2 + z^2}$ $\tan\alpha = y/x$

A5. Operations of Differentiation

Operation	Symbol	Rectangular coordinates, x, y, z	Cylindrical coordinates, r, α, z	Spherical coordinates, R, θ, α
grad φ	$\nabla \varphi$	$\mathbf{e}_x \frac{\partial \varphi}{\partial x} + \mathbf{e}_y \frac{\partial \varphi}{\partial y} + \mathbf{e}_z \frac{\partial \varphi}{\partial z}$	$\mathbf{e}_r \frac{\partial \varphi}{\partial r} + \mathbf{e}_\alpha \frac{1}{r} \frac{\partial \varphi}{\partial \alpha} + \mathbf{e}_z \frac{\partial \varphi}{\partial z}$	$\mathbf{e}_R \frac{\partial \varphi}{\partial R} + \mathbf{e}_\theta \frac{1}{R} \frac{\partial \varphi}{\partial \theta} + \mathbf{e}_\alpha \frac{1}{R \sin \theta} \frac{\partial \varphi}{\partial \alpha}$
div $\mathbf{F}$	$\nabla \mathbf{F}$	$\frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}$	$\frac{1}{r} \frac{\partial (r F_r)}{\partial r} + \frac{1}{r} \frac{\partial F_\alpha}{\partial \alpha} + \frac{\partial F_z}{\partial z}$	$\frac{1}{R^2} \frac{\partial (R^2 F_R)}{\partial R} + \frac{1}{R \sin \theta} \frac{\partial (\sin \theta F_\theta)}{\partial \theta} + \frac{1}{R \sin \theta} \frac{\partial F_\alpha}{\partial \alpha}$
curl $\mathbf{F}$	$\nabla \times \mathbf{F}$	$\begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix}$	$\begin{vmatrix} \frac{\mathbf{e}_r}{r} & \mathbf{e}_\alpha & \frac{\mathbf{e}_z}{r} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \alpha} & \frac{\partial}{\partial z} \\ F_r & r F_\alpha & F_z \end{vmatrix}$	$\begin{vmatrix} \frac{\mathbf{e}_R}{R^2 \sin \theta} & \frac{\mathbf{e}_\theta}{R \sin \theta} & \frac{\mathbf{e}_\alpha}{R} \\ \frac{\partial}{\partial R} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \alpha} \\ F_R & R F_\theta & R \sin \theta F_\alpha \end{vmatrix}$
div grad φ	$\Delta \varphi = \nabla^2 \varphi$	$\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} + \frac{\partial^2 \varphi}{\partial z^2}$	$\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial \varphi}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 \varphi}{\partial \alpha^2} + \frac{\partial^2 \varphi}{\partial z^2}$	$\frac{1}{R} \frac{\partial^2 (R \varphi)}{\partial R^2} + \frac{1}{R^2 \sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial \varphi}{\partial \theta} \right) + \frac{1}{R^2 \sin^2 \theta} \frac{\partial^2 \varphi}{\partial \alpha^2}$

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